MAT123 MATHEMATICS I

Lecture 02: Functions and Their Graphs

Outline

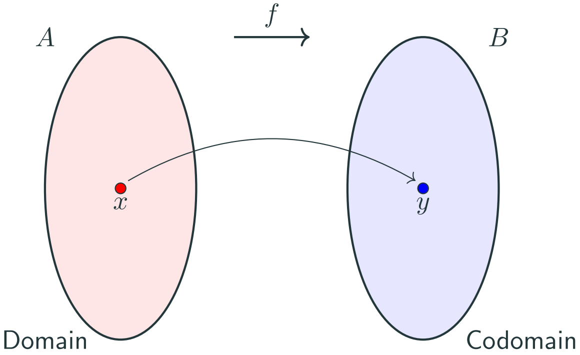
Functions

Polynomials and Rational Functions

Functions

Functions

A **function** is a relation that assigns exactly one output for each input. We denote a function by $f : A \to B$, where A is the domain and B is the codomain.



$$\boldsymbol{y = f(x)}$$

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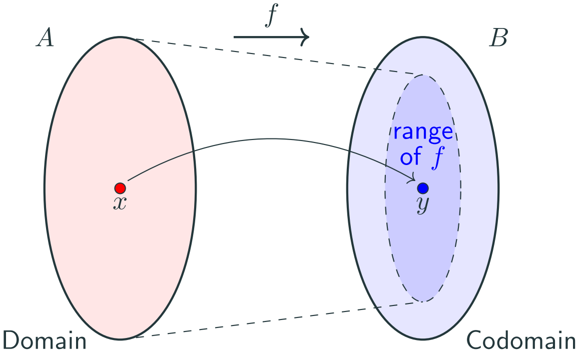

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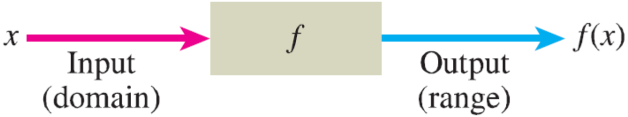
Functions



A function machine

Functions

$$y = f(x)$$

Functions

independent variable

$$y = f(x)$$

dependent variable

↑

$$y = f(x)$$

Functions

A function can be represented in various ways:

- **Algebraic form:** $f(x) = x^2 + 3x + 2$
  $y = x^2 + 3x + 2$
- **Table of values:**

x	$f(x)$
-2	0
-1	0
0	2
1	6
2	12

- **Graph:** A visual representation of the function in the Cartesian plane.

The Domain convention

The domain of a function is the set of all possible input values (independent variables).

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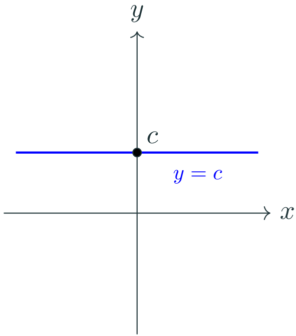
Example

The domain of the function $f(x) = \sqrt{1 - x^2}$ is the set of all x such that $1 - x^2 \geq 0$. This means $-1 \leq x \leq 1$. Therefore, the domain is $[-1, 1]$.

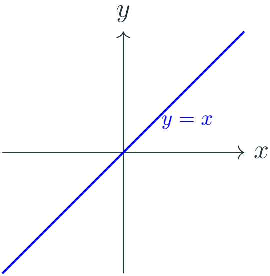
A picture is worth a thousand words.

Definition.

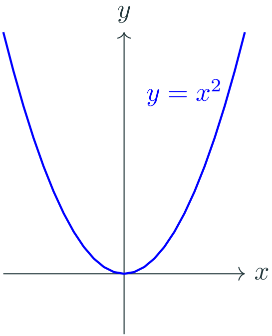
A **graph** of a function f is the set of all points $(x, f(x))$ in the Cartesian plane, where x is in the domain of f.



The graph of a constant function $f(x) = c$



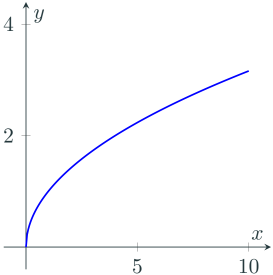
The graph of $f(x) = x$



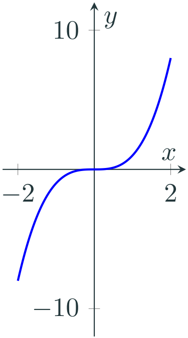
The graph of $f(x) = x^2$

Functions

Graphs of Functions



The graph of
$f(x) = \sqrt{x}$



The graph of
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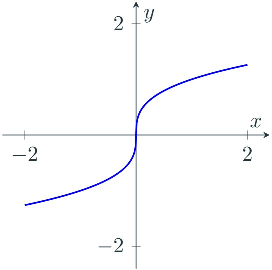
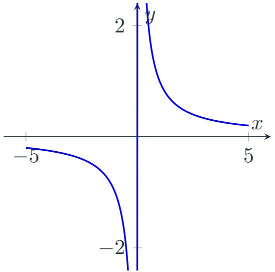


The graph of
$f(x) = x^{\frac{1}{3}}$



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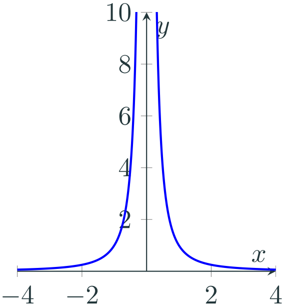

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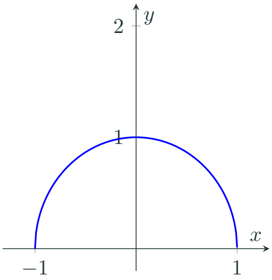
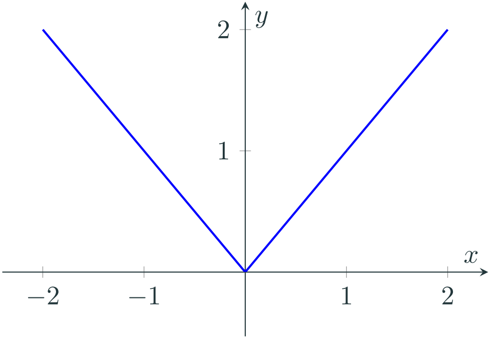


The graph of
$f(x) = \sqrt{1 - x^2}$



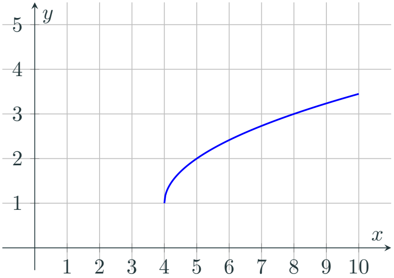
The graph of $f(x) = |x|$

Example

Sketch the graph of $y = 1 + \sqrt{x-4}$.

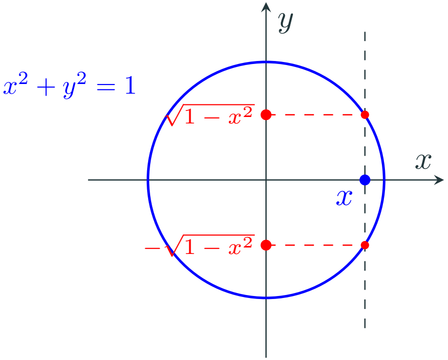
Solution

The function $y = 1 + \sqrt{x-4}$ *is defined for* $x \geq 4$*. It can be obtained by shifting the graph of* $y = \sqrt{x}$ *to the right by 4 units and up by 1 unit. The graph is shown below:*



Example

Not every curve is the graph of a function. Consider the curve defined by the equation $x^2 + y^2 = 1$. This is not a graph of a function because for some values of x, there are two corresponding values of y (one positive and one negative).



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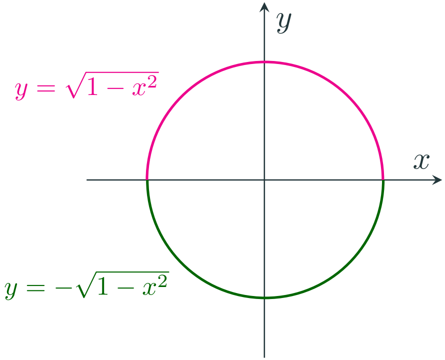

Definition.

- A function f is called **even** if for all x in the domain of f, $f(-x) = f(x)$. The graph of an even function is symmetric with respect to the y-axis.

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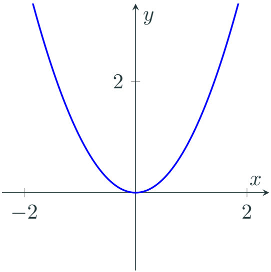
Example

The function $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)$.

The function $f(x) = x^3$ is odd because $f(-x) = (-x)^3 = -x^3 = -f(x)$.



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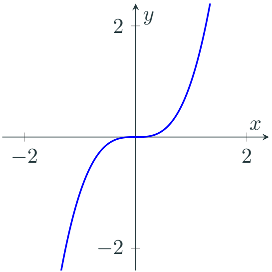
Symmetric with respect to the y-axis.



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Example

Describe and sketch the graph of $y = \sqrt{2 - x} - 3$.

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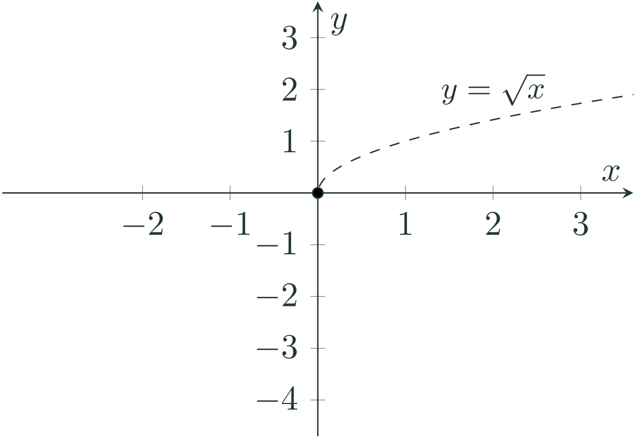
Solution



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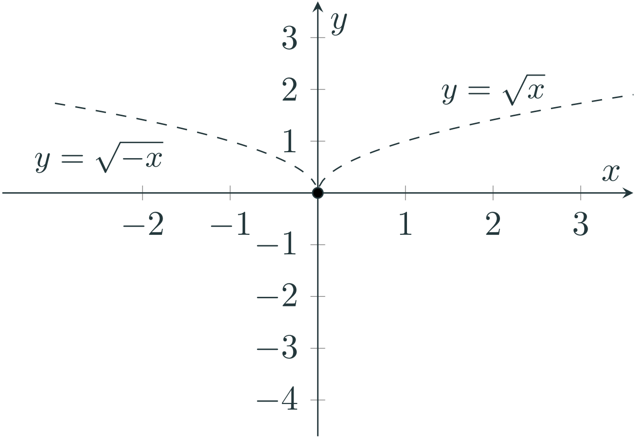

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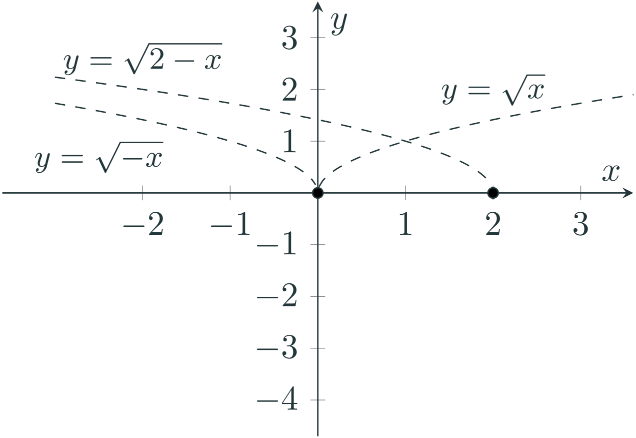

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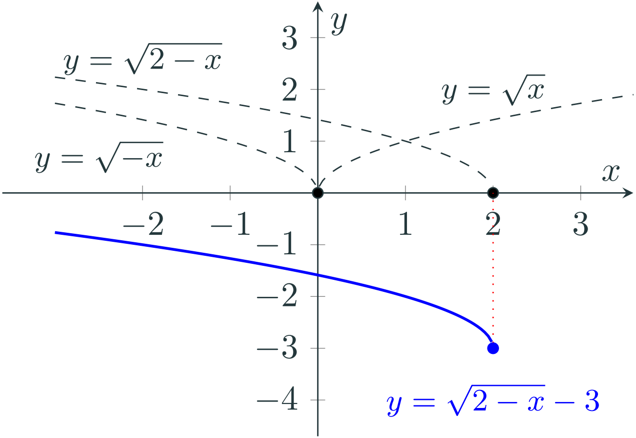

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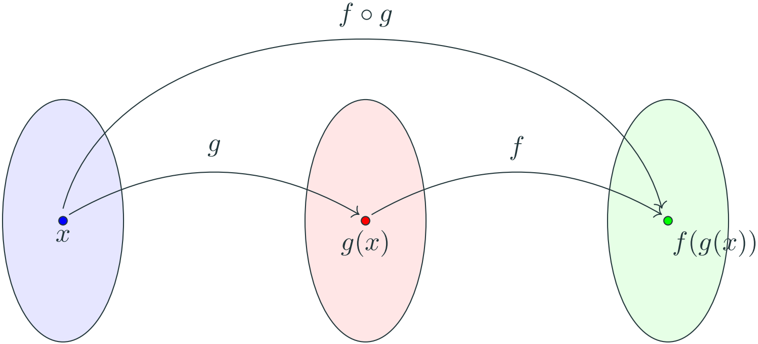
- The **constant multiple** of f by a constant c is defined by $(cf)(x) = c \cdot f(x)$.

Functions

Composition of Functions

Definition.

Let f and g be functions. The **composition** of f and g, denoted by $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$.



Functions

Composition of Functions

Function	Formula	Domain
f	$f(x) = \sqrt{x}$	$[0, \infty)$
g	$g(x) = x + 1$	$\mathbb{R}$
$f \circ g$	$(f \circ g)(x) = f(g(x)) = f(x+1) = \sqrt{x+1}$	$[-1, \infty)$
$g \circ f$	$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 1$	$[0, \infty)$
$f \circ f$	$(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
$g \circ g$	$(g \circ g)(x) = g(g(x)) = g(x+1) = g(x) + 1 = x + 2$	$\mathbb{R}$

Example

For $h(x) = \frac{1-x}{1+x}$, $h \circ h(x) = x$. But the domain of the function $h \circ h$ is not $\mathbb{R}$, it is $\mathbb{R} \setminus \{-1\}$.

Definition.

A **piecewise defined function** is a function that is defined by different expressions on different parts of its domain.

Example

The function f defined by

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$$

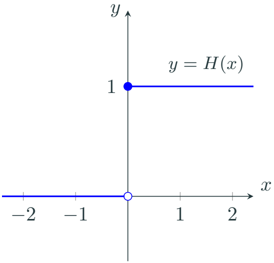
is a piecewise defined function.

Example (The Heaviside function)

The Heaviside function $H(x)$ is defined as:

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

It is often used in control theory and signal processing. The graph of the Heaviside function is a step function that jumps from 0 to 1 at $x = 0$.

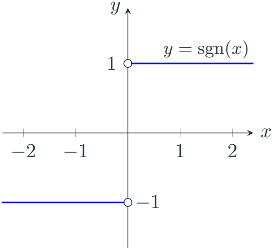


Example (The sign function)

The sign function $sgn(x)$ is defined as:

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0. \end{cases}$$

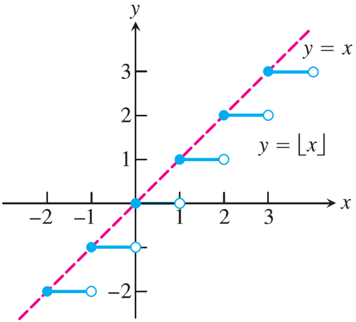
It indicates the sign of a real number. The graph of the sign function is as follows:



Example (The floor function)

The floor function $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x. It can be expressed as a piecewise function:

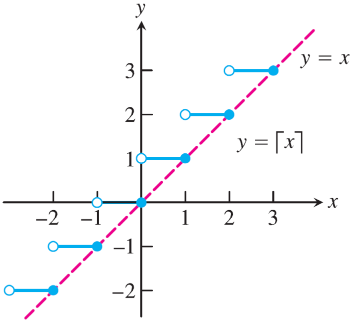
$$\lfloor x \rfloor = \begin{cases} n & \text{if } n \leq x < n+1, \quad n \in \mathbb{Z} \end{cases}$$



Example (The ceiling function)

The ceiling function $\lceil x \rceil$ is defined as the smallest integer greater than or equal to x. It can be expressed as a piecewise function:

$$\lceil x \rceil = \begin{cases} n & \text{if } n-1 < x \leq n, \quad n \in \mathbb{Z} \end{cases}$$



Polynomials and Rational Functions

Definition.

A **polynomial function** is a function of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_0$ are constants (coefficients), n is a non-negative integer, and $a_n \neq 0$.

The highest power n of x in the polynomial is called the **degree** of the polynomial. We write $\deg(f) = n$.

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Remark.

The degree of the zero polynomial (i.e., the polynomial that is identically zero) is not defined.

Polynomials and Rational Functions

Definition.

A **rational function** is a function of the form:

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

Theorem. *Division Algorithm for Polynomials*

Let $p(x)$ and $q(x)$ be polynomials with $q(x) \neq 0$. Then there exist unique polynomials $d(x)$ and $r(x)$ such that:

$$p(x) = d(x) \cdot q(x) + r(x)$$

where either $r(x) = 0$ or $\deg(r) < \deg(q)$.

Thus, we may write:

$$\frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}.$$

Example

Write the division algorithm for $\dfrac{2x^3 - 5x^2 + 3x + 4}{x^2 + 1}$.

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Solution

We divide $2x^3 - 5x^2 + 3x + 4$ *by* $x^2 + 1$*:*

$2x^3 - 5x^2 + 3x + 4$	$x^2 + 1$
$\vdots$	$2x - 5$
$-$	
$x + 9$	

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Thus, we have:

$$\frac{2x^3 - 5x^2 + 3x + 4}{x^2 + 1} = 2x - 5 + \frac{x + 9}{x^2 + 1}.$$

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Example

The polynomial $p(x) = x^2 - 5x + 6$ can be factored as:

$$p(x) = (x - 2)(x - 3)$$

The roots of this polynomial are $r_1 = 2$ and $r_2 = 3$.

Theorem. *Fundamental Theorem of Algebra*

Every non-constant polynomial function $p(x)$ of degree $n \geq 1$ has exactly n roots in the complex numbers, counting multiplicities.

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Example

The polynomial $p(x) = x^3 - 6x^2 + 11x - 6$ has roots 1, 2, and 3. It can be factored as:

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Example

What is the degree of $P(x) = x^3(x^2 + 2x + 5)^2$? What are the roots of $P(x)$? What is the multiplicity of each root?

Theorem. *Factor Theorem*

Let $p(x)$ be a polynomial. If r is a root of $p(x)$, then $(x - r)$ is a factor of $p(x)$.

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Theorem. *Rational Root Theorem*

If $p(x)$ is a polynomial with integer coefficients, and $r = a/b$ is a rational root (in lowest terms), then:

- a divides the constant term $p(0)$.
- b divides the leading coefficient of $p(x)$.

Example

Factor the polynomial $p(x) = x^3 - 4x^2 + 5x - 2$.

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Solution

To factor the polynomial, we can use the Rational Root Theorem to find possible rational roots. Testing $x = 1$:

$$p(1) = 1^3 - 4 \cdot 1^2 + 5 \cdot 1 - 2 = 0.$$

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Thus, $x = 1$ *is a root. We can then perform polynomial long division to factor* $p(x)$*:* $p(x) = (x - 1)(x^2 - 3x + 2)$*.The quadratic can be factored further:* $x^2 - 3x + 2 = (x - 1)(x - 2)$*.Therefore, the complete factorization is:* $p(x) = (x - 1)^2(x - 2)$*.*

Thus, the roots are $x = 1$ *(with multiplicity 2) and* $x = 2$*.*

- The **difference of squares**:

$$a^2 - b^2 = (a - b)(a + b).$$

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- The **sum of cubes** and **difference of cubes** can be factored as follows:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2),$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

- The **difference of squares**:

$$a^2 - b^2 = (a - b)(a + b).$$

- The **sum of cubes** and **difference of cubes** can be factored as follows:

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$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

- More generally, a difference of n-th powers of a and b can be factored as:

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}),$$

where n is a positive integer.

Remark

Moreover, if $n > 0$ is an odd integer, then

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots + b^{n-1}).$$

Example

Find the roots of the following polynomials:

(a) $P(x) = x^3 - x^2 - 4x + 4$, (b) $P(x) = x^4 - 3x^2 - 4$,

(c) $P(x) = x^5 - x^4 - x^2 + x$.

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Solution

(a) We can factor $P(x)$ as follows:

$$P(x) = x^3 - x^2 - 4x + 4 = (x-1)(x-2)(x+2).$$

The roots are $x = 1$, $x = 2$ and $x = -2$ (with multiplicity 1).

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Find the roots of the following polynomials:

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Solution

(b) We can factor $P(x)$ as follows:

$$P(x) = x^4 - 3x^2 - 4 = (x^2 - 4)(x^2 + 1) = (x - 2)(x + 2)(x^2 + 1).$$

The roots are $x = 2$ and $x = -2$ (both with multiplicity 1), and $x = i$ and $x = -i$ (both with multiplicity 1).

Example

Find the roots of the following polynomials:

(a) $P(x) = x^3 - x^2 - 4x + 4$, (b) $P(x) = x^4 - 3x^2 - 4$,
(c) $P(x) = x^5 - x^4 - x^2 + x$.

Solution
(c) We can factor $P(x)$ as follows:

$$P(x) = x^5 - x^4 - x^2 + x = (x^2 + x + 1)(x - 1)^2 x.$$

The roots are $x = 1$ (with multiplicity 2), $x = 0$ (with multiplicity 1), and the roots of $x^2 + x + 1 = 0$, which are complex roots $x = \frac{-1 \pm i\sqrt{3}}{2}$ (both with multiplicity 1).