

MAT123 MATHEMATICS I

Lecture 04: Inverse Trigonometric Functions, Hyperbolic Functions

The Inverse Trigonometric Functions

Inverse Sine Function

Inverse Tangent Function

Other Inverse Trigonometric Functions

Hyperbolic Functions

The Inverse Trigonometric Functions

Inverse Trigonometric Functions

- To identify a function as having an inverse, it must be one-to-one.

Inverse Trigonometric Functions

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Inverse Trigonometric Functions

- To identify a function as having an inverse, it must be one-to-one.
- The title may be confusing, as the trigonometric functions are not one-to-one.
- To define the inverse, we restrict the domain of the trigonometric functions so that they become one-to-one while having largest possible domains.

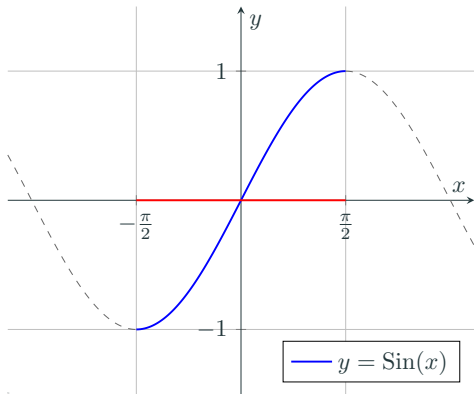
Inverse Trigonometric Functions

Inverse Sine Function

Definition (*The restricted sine function*)

The restricted sine function Sin is defined as:

$$\text{Sin} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1], \quad x \mapsto \sin(x)$$



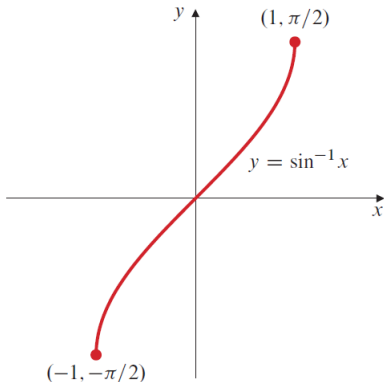
Inverse Trigonometric Functions

Inverse Sine Function

Definition (*The inverse sine function*)

The inverse sine function Sin^{-1} is defined as:

$$\text{Sin}^{-1} : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad y \mapsto x \text{ such that } \sin(x) = y$$



Inverse Trigonometric Functions

Inverse Sine Function

- $\sin^{-1}(\sin x) = \arcsin(\sin x) = x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Inverse Trigonometric Functions

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- $\sin^{-1}(\sin x) = \arcsin(\sin x) = x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
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Example

Find (a) $\sin(\sin^{-1} 0.7)$, (b) $\sin^{-1}(\sin 0.3)$, (c) $\sin^{-1}(\sin \frac{4\pi}{5})$, and (d) $\cos(\sin^{-1} 0.6)$.

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Solution

Inverse Trigonometric Functions

Inverse Sine Function

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Find (a) $\sin(\sin^{-1} 0.7)$, (b) $\sin^{-1}(\sin 0.3)$, (c) $\sin^{-1}(\sin \frac{4\pi}{5})$, and (d) $\cos(\sin^{-1} 0.6)$.

Solution

(a) $\sin(\sin^{-1} 0.7) = 0.7$

Inverse Trigonometric Functions

Inverse Sine Function

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Example

Find (a) $\sin(\sin^{-1} 0.7)$, (b) $\sin^{-1}(\sin 0.3)$, (c) $\sin^{-1}(\sin \frac{4\pi}{5})$, and (d) $\cos(\sin^{-1} 0.6)$.

Solution

(a) $\sin(\sin^{-1} 0.7) = 0.7$

(b) $\sin^{-1}(\sin 0.3) = 0.3$

Inverse Trigonometric Functions

Inverse Sine Function

- $\sin^{-1}(\sin x) = \arcsin(\sin x) = x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
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Example

Find (a) $\sin(\sin^{-1} 0.7)$, (b) $\sin^{-1}(\sin 0.3)$, (c) $\sin^{-1}(\sin \frac{4\pi}{5})$, and (d) $\cos(\sin^{-1} 0.6)$.

Solution

$$(a) \sin(\sin^{-1} 0.7) = 0.7$$

$$(b) \sin^{-1}(\sin 0.3) = 0.3$$

$$(c) \sin^{-1}(\sin \frac{4\pi}{5}) = \frac{\pi}{5}$$

Inverse Trigonometric Functions

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Example

Find (a) $\sin(\sin^{-1} 0.7)$, (b) $\sin^{-1}(\sin 0.3)$, (c) $\sin^{-1}(\sin \frac{4\pi}{5})$, and (d) $\cos(\sin^{-1} 0.6)$.

Solution

$$(a) \sin(\sin^{-1} 0.7) = 0.7$$

$$(b) \sin^{-1}(\sin 0.3) = 0.3$$

$$(c) \sin^{-1}(\sin \frac{4\pi}{5}) = \frac{\pi}{5}$$

$$(d) \cos(\sin^{-1} 0.6) = \sqrt{1 - 0.6^2} = \sqrt{0.64} = 0.8$$

Inverse Trigonometric Functions

Inverse Sine Function

Example

Simplify the expression $\tan(\sin^{-1} x)$.

Inverse Trigonometric Functions

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Simplify the expression $\tan(\sin^{-1} x)$.

Solution

We know that $\sin(\sin^{-1} x) = x$ for $x \in [-1, 1]$.

Inverse Trigonometric Functions

Inverse Sine Function

Example

Simplify the expression $\tan(\sin^{-1} x)$.

Solution

We know that $\sin(\sin^{-1} x) = x$ for $x \in [-1, 1]$. First, we find $\cos(\sin^{-1} x)$ using the Pythagorean identity:

$$\cos(\sin^{-1} x) = \sqrt{1 - [\sin(\sin^{-1} x)]^2} = \sqrt{1 - x^2}$$

Inverse Trigonometric Functions

Inverse Sine Function

Example

Simplify the expression $\tan(\sin^{-1} x)$.

Solution

We know that $\sin(\sin^{-1} x) = x$ for $x \in [-1, 1]$. First, we find $\cos(\sin^{-1} x)$ using the Pythagorean identity:

$$\cos(\sin^{-1} x) = \sqrt{1 - [\sin(\sin^{-1} x)]^2} = \sqrt{1 - x^2}$$

Then, we can express $\tan(\sin^{-1} x)$ as:

$$\tan(\sin^{-1} x) = \frac{\sin(\sin^{-1} x)}{\cos(\sin^{-1} x)} = \frac{x}{\sqrt{1 - x^2}}$$

for $x \in [-1, 1]$.

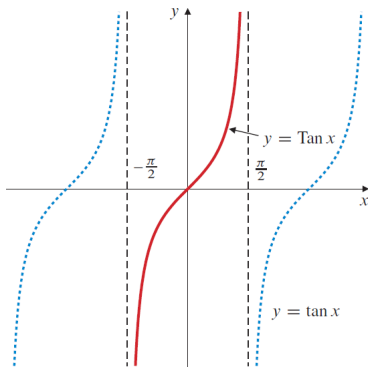
Inverse Trigonometric Functions

Inverse Tangent Function

Definition (*The restricted tangent function*)

The restricted tangent function Tan is defined as:

$$\text{Tan} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}, \quad x \mapsto \tan(x)$$



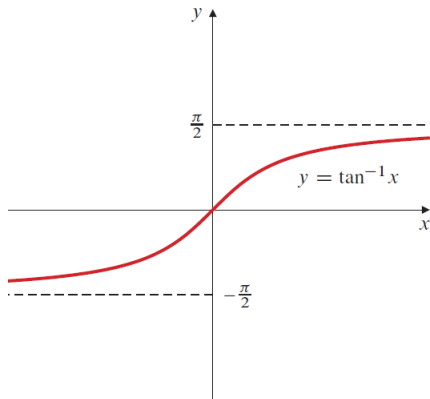
Inverse Trigonometric Functions

Inverse Tangent Function

Definition (*The inverse tangent function*)

The inverse tangent function Tan^{-1} is defined as:

$$\text{Tan}^{-1} : \mathbb{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad y \mapsto x \text{ such that } \tan(x) = y$$



Inverse Trigonometric Functions

Inverse Tangent Function

- $y = \tan^{-1} x \iff x = \text{Tan } y \iff x = \tan y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$
- $\tan^{-1}(\text{Tan } x) = \arctan(\text{Tan } x) = x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $\text{Tan}(\tan^{-1} y) = \text{Tan}(\arctan y) = y$ for $y \in \mathbb{R}$

Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

- The function $\cos x$ is one-to-one on the interval $[0, \pi]$.

Inverse Trigonometric Functions

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- The function $\cos x$ is one-to-one on the interval $[0, \pi]$.
- The inverse cosine function \cos^{-1} is defined as:

$$y = \cos^{-1} x \iff x = \cos y \text{ and } 0 \leq y \leq \pi$$

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- The inverse cosine function \cos^{-1} is defined as:

$$y = \cos^{-1} x \iff x = \cos y \text{ and } 0 \leq y \leq \pi$$

- Observe that

$$\begin{aligned} y = \cos^{-1} x &\iff x = \sin\left(\frac{\pi}{2} - y\right) \text{ and } 0 \leq y \leq \pi \\ &\iff \sin^{-1} x = \frac{\pi}{2} - y = \frac{\pi}{2} - \cos^{-1} x, \end{aligned}$$

whence

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x.$$

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whence

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x.$$

- $\cos^{-1}(\cos x) = \arccos(\cos x) = x$ for $x \in [0, \pi]$.

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whence

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x.$$

- $\cos^{-1}(\cos x) = \arccos(\cos x) = x$ for $x \in [0, \pi]$.
- $\cos(\cos^{-1} y) = \cos(\arccos y) = y$ for $y \in [-1, 1]$.

Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

- The function \sec is one-to-one on the interval $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

- The function \sec is one-to-one on the interval $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
- The inverse secant function \sec^{-1} is defined as:

$$y = \sec^{-1} x \iff x = \sec y \text{ and } 0 \leq y < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < y \leq \pi$$

Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

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$$y = \sec^{-1} x \iff x = \sec y \text{ and } 0 \leq y < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < y \leq \pi$$

- Observe that

$$\begin{aligned} y = \sec^{-1} x &\iff x = \frac{1}{\cos y} \text{ and } 0 \leq y < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < y \leq \pi \\ &\iff \cos^{-1} \left(\frac{1}{x} \right) = y = \sec^{-1} x, \end{aligned}$$

whence

$$\sec^{-1} x = \cos^{-1} \left(\frac{1}{x} \right).$$

Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

- The function \sec is one-to-one on the interval $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
- The inverse secant function \sec^{-1} is defined as:

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- Observe that

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whence

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right).$$

- $\sec^{-1}(\sec x) = \operatorname{arcsec}(\sec x) = x$ for $x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

- The function \sec is one-to-one on the interval $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
- The inverse secant function \sec^{-1} is defined as:

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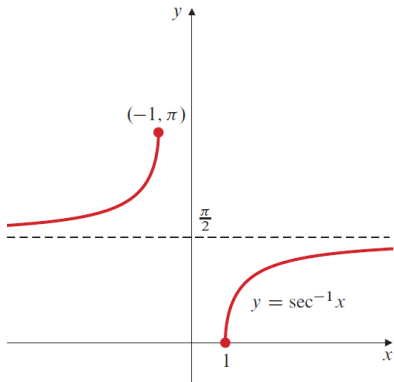
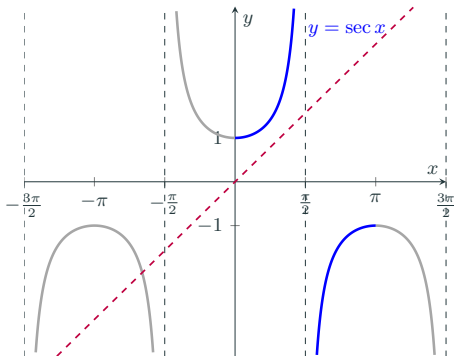
whence

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right).$$

- $\sec^{-1}(\sec x) = \operatorname{arcsec}(\sec x) = x$ for $x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
- $\sec(\sec^{-1} y) = \sec(\operatorname{arcsec} y) = y$ for $y \in (-\infty, -1] \cup [1, \infty)$.

Inverse Trigonometric Functions

Other Inverse Trigonometric Functions



Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

The Inverse cosecant Function

- The function \csc^{-1} is one-to-one on the interval $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.

Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

The Inverse cosecant Function

- The function \csc is one-to-one on the interval $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.
- The inverse cosecant function \csc^{-1} is defined as:

$$y = \csc^{-1} x \iff x = \csc y \text{ and } -\frac{\pi}{2} \leq y < 0 \text{ or } 0 < y \leq \frac{\pi}{2}$$

Inverse Trigonometric Functions

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- The inverse cosecant function \csc^{-1} is defined as:

$$y = \csc^{-1} x \iff x = \csc y \text{ and } -\frac{\pi}{2} \leq y < 0 \text{ or } 0 < y \leq \frac{\pi}{2}$$

- Observe that

$$\begin{aligned} y = \csc^{-1} x &\iff x = \frac{1}{\sin y} \text{ and } -\frac{\pi}{2} \leq y < 0 \text{ or } 0 < y \leq \frac{\pi}{2} \\ &\iff \sin^{-1} \left(\frac{1}{x} \right) = y = \csc^{-1} x, \end{aligned}$$

whence

$$\csc^{-1} x = \sin^{-1} \left(\frac{1}{x} \right).$$

Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

The Inverse Cotangent Function

- The function \cot is one-to-one on the interval $(0, \pi)$.

Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

The Inverse Cotangent Function

- The function \cot is one-to-one on the interval $(0, \pi)$.
- The inverse cotangent function \cot^{-1} is defined as:

$$y = \cot^{-1} x \iff x = \cot y \text{ and } 0 < y < \pi$$

Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

The Inverse Cotangent Function

- The function \cot is one-to-one on the interval $(0, \pi)$.
- The inverse cotangent function \cot^{-1} is defined as:

$$y = \cot^{-1} x \iff x = \cot y \text{ and } 0 < y < \pi$$

- Observe that

$$y = \cot^{-1} x \iff x = \frac{\cos y}{\sin y} \text{ and } 0 < y < \pi$$
$$\iff y \neq \frac{\pi}{2} \iff \tan^{-1}\left(\frac{1}{x}\right) = y = \cot^{-1} x$$

whence

$$\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right), \text{ where } y \in (0, \pi) - \left\{\frac{\pi}{2}\right\}.$$

Hyperbolic Functions

Hyperbolic Functions

- The hyperbolic sine function is defined as:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic Functions

- The hyperbolic sine function is defined as:

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- The hyperbolic cosine function is defined as:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic Functions

- The hyperbolic sine function is defined as:

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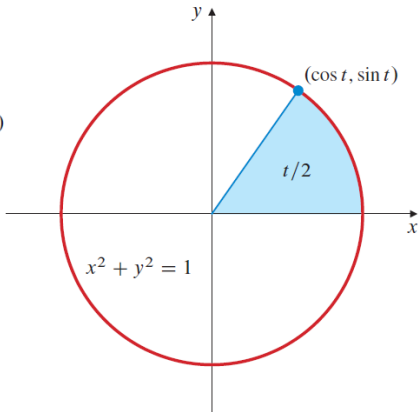
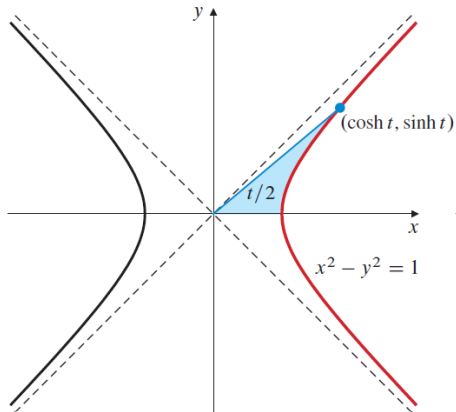
- The hyperbolic cosine function is defined as:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

- The hyperbolic tangent function is defined as:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic Functions



Hyperbolic Functions

