

MAT123 MATHEMATICS I

Lecture 05: Limits and Continuity

Limits

Informal Definition of Limits

One-Sided Limits

Calculating Limits

The Squeeze Theorem

Limits

Limits

Average Velocity and Instantaneous Velocity

The average velocity of a moving object over an interval of time $[a, b]$ is given by:

$$v_{\text{avg}} = \frac{\text{change in position}}{\text{change in time}} = \frac{s(b) - s(a)}{b - a},$$

where $s(t)$ is the position function.

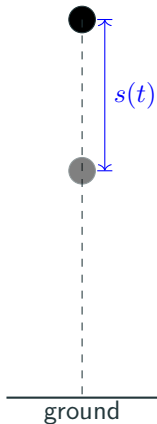
Limits

Instantaneous Velocity

Example (Average velocity of a falling rock)

Let $s(t) = 4.9t^2$ be the displacement of a rock falling from a height at time t . The average velocity during the first 2 seconds is:

$$\begin{aligned}v_{\text{avg}} &= \frac{s(2) - s(0)}{2 - 0} = \frac{(4.9(2)^2) - (4.9(0)^2)}{2 - 0} \\ &= \frac{19.6}{2} = 9.8\end{aligned}$$

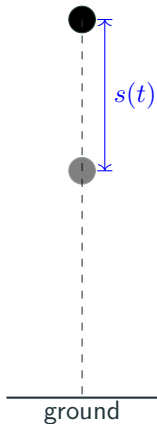


Limits

Instantaneous Velocity

Example (Instantaneous velocity of a falling rock)

How fast is the rock falling at $t = 2$ seconds?



Limits

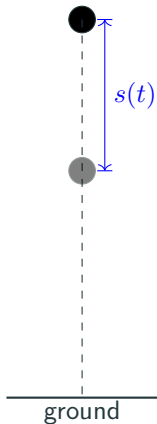
Instantaneous Velocity

Example (Instantaneous velocity of a falling rock)

How fast is the rock falling at $t = 2$ seconds?

The average velocity over the interval $[2, 2 + h]$ is:

$$\begin{aligned}v_{\text{avg}} &= \frac{s(2+h) - s(2)}{(2+h) - 2} \\&= \frac{(4.9(2+h)^2) - (4.9(2)^2)}{h} \\&= \frac{4.9((2+h)^2 - (2)^2)}{h} \\&= \frac{4.9(4 + 4h + h^2 - 4)}{h} \\&= \frac{4.9(4h + h^2)}{h} = 4.9(4 + h).\end{aligned}$$



Limits

Instantaneous Velocity

Example (Instantaneous velocity of a falling rock)

How fast is the rock falling at $t = 2$ seconds?

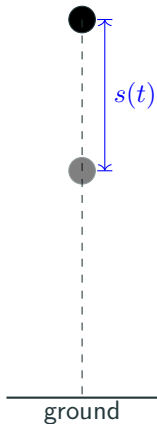
For several values of h :

$$h = 0.1 : v_{\text{avg}} = 4.9(4 + 0.1) = 20.0900$$

$$h = 0.01 : v_{\text{avg}} = 4.9(4 + 0.01) = 19.6490$$

$$h = 0.001 : v_{\text{avg}} = 4.9(4 + 0.001) = 19.6049$$

$$h = 0.0001 : v_{\text{avg}} = 4.9(4 + 0.0001) = 19.6005$$



Limits

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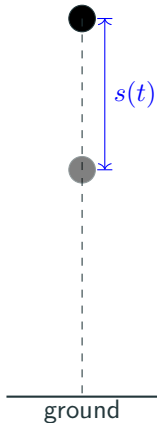
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As h approaches 0, the average velocity approaches:
 $4.9(4 + 0) = 19.6$.



Limits

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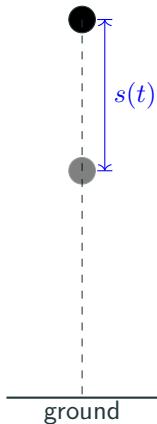
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As h approaches 0, the average velocity approaches:
 $4.9(4 + 0) = 19.6$.

Notation:

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{4.9(2+\Delta t)^2 - 4.9(2)^2}{h} = 19.6.$$



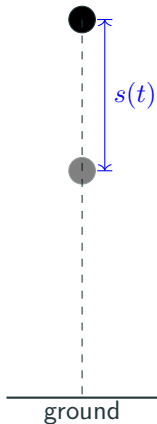
Limits

Instantaneous Velocity

More generally

The instantaneous velocity at time t is defined as:

$$\begin{aligned}\lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} &= \lim_{\Delta t \rightarrow 0} \frac{4.9(t + \Delta t)^2 - 4.9t^2}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{4.9((t + \Delta t)^2 - t^2)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{4.9(2t\Delta t + \Delta t^2)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} 4.9(2t + \Delta t) \\ &= 4.9(2t + 0) = 9.8t.\end{aligned}$$



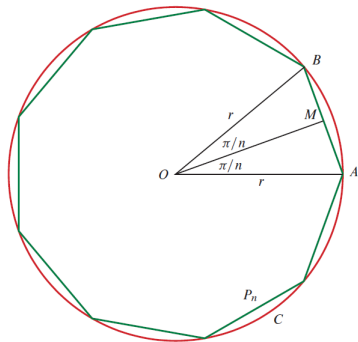
Limits

The area of a circle

The area of a circle can be approximated by inscribing polygons within the circle. As the number of sides of the polygon increases, the polygon looks more and more like the circle.

Limits

Area of a circle



P_n : the perimeter of the inscribed polygon with n sides

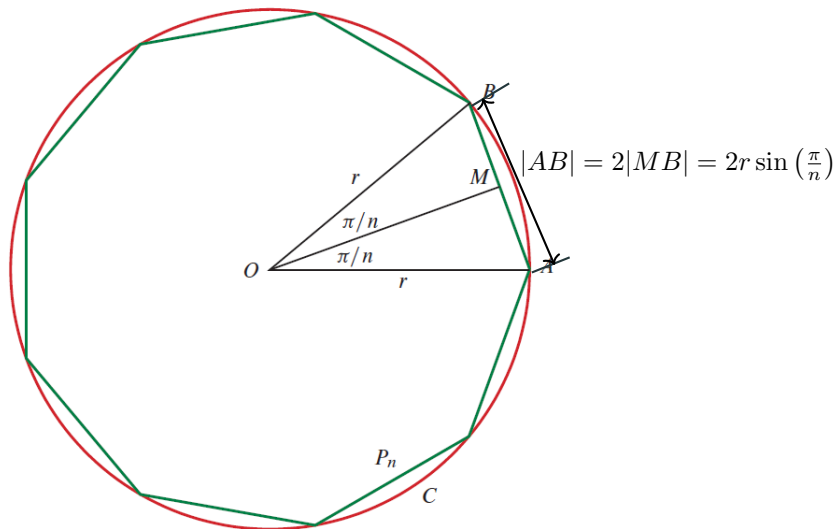
A_n : the area of the inscribed polygon with n sides

A : the area of the circle

C : the circumference of the circle

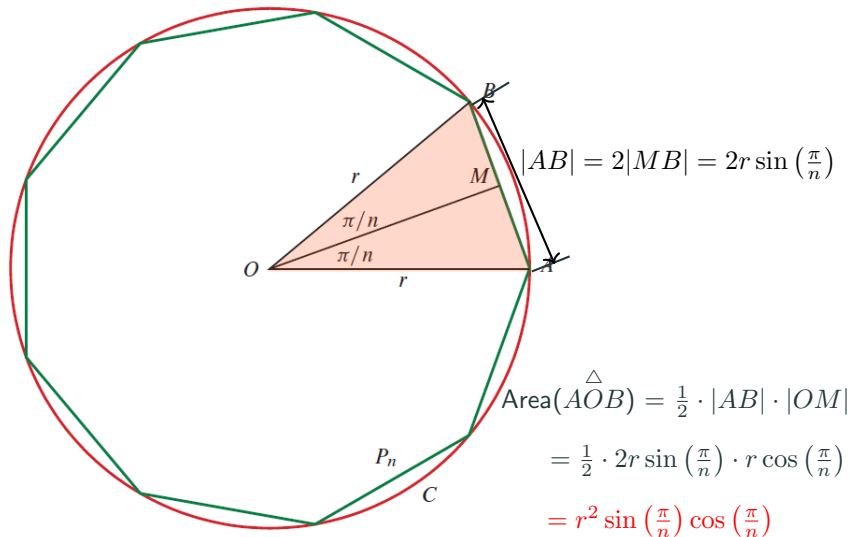
Limits

Area of a circle



Limits

Area of a circle

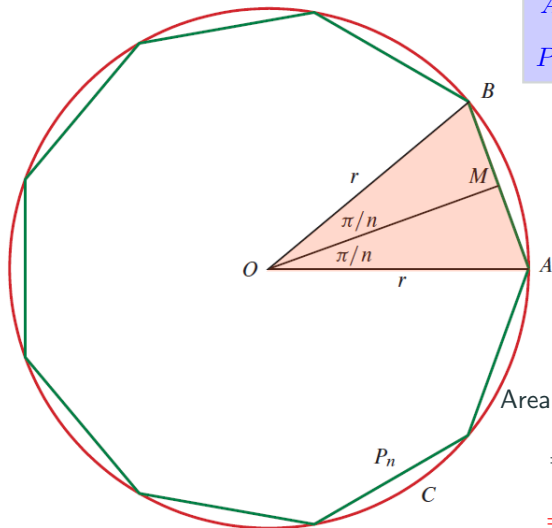


Limits

Area of a circle

$$A_n = n \cdot r^2 \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)$$

$$P_n = 2nr \sin\left(\frac{\pi}{n}\right)$$



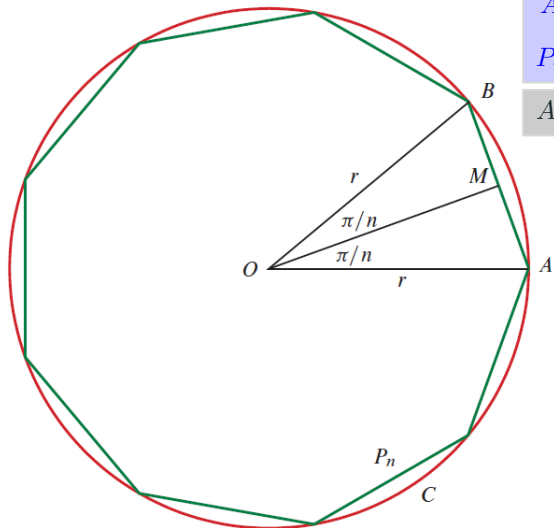
$$\text{Area}(\triangle AOB) = \frac{1}{2} \cdot |AB| \cdot |OM|$$

$$= \frac{1}{2} \cdot 2r \sin\left(\frac{\pi}{n}\right) \cdot r \cos\left(\frac{\pi}{n}\right)$$

$$= r^2 \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)$$

Limits

Area of a circle



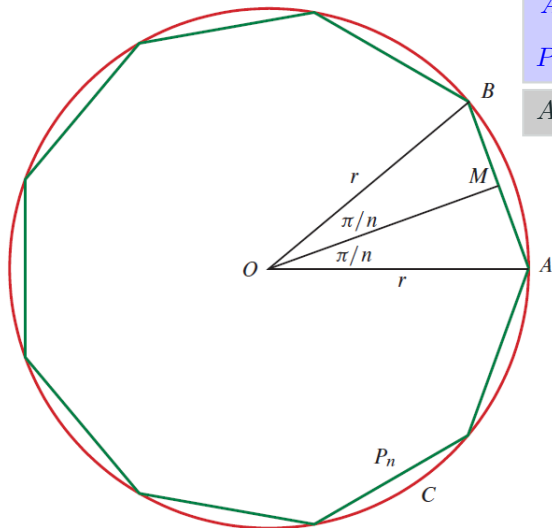
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As n grows
large

$$\frac{\pi}{n} \rightarrow 0$$

$$\cos\left(\frac{\pi}{n}\right) = \frac{|OM|}{|OA|} \rightarrow 1$$

$$P_n \rightarrow C = 2\pi r$$

$$A_n \rightarrow A = \pi r^2$$

Example

Describe the behaviour of the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

near $x = 1$.

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

Example

What happens to the function $f(x) = (1 + x^2)^{1/x^2}$ as x approaches zero?

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x	$f(x)$
± 1.0	2.000000000
± 0.1	2.704813829
± 0.01	2.718145927
± 0.001	2.718280469
± 0.0001	2.718281815
± 0.00001	1.000000000

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$$\lim_{x \rightarrow 0} f(x) = 2.7182818 \dots = e$$

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± 0.00001	1.000000000

$\lim_{x \rightarrow 0} f(x) = 2.7182818 \dots = e$

→ round-off error

An informal definition of limit

If $f(x)$ is defined for all x in some open interval around a , except possibly at a itself, then we say that the limit of $f(x)$ as x approaches a is L , and write:

$$\lim_{x \rightarrow a} f(x) = L,$$

if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to a , but not equal to a .

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Example

- $\lim_{x \rightarrow a} x = a$

An informal definition of limit

If $f(x)$ is defined for all x in some open interval around a , except possibly at a itself, then we say that the limit of $f(x)$ as x approaches a is L , and write:

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Example

- $\lim_{x \rightarrow a} x = a$
- $\lim_{x \rightarrow a} c = c$

Example

Let

$$g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$$

Then $\lim_{x \rightarrow 2} g(x) = 2$ because we can make $g(x)$ as close to 2 as we want by taking x close enough to 2, but not equal to 2.

In other words,

$$\lim_{x \rightarrow 2} g(x) = \lim_{x \rightarrow 2} x = 2$$

although $g(2) = 1$.

Limits

One-Sided Limits

Informal definition of one-sided limits

We say that the limit of $f(x)$ as x approaches a from the left is L and write

$$\lim_{x \rightarrow a^-} f(x) = L,$$

if we can make $f(x)$ as close as we want to L by taking x close enough to a from the left (i.e., $x < a$).

Limits

One-Sided Limits

Informal definition of one-sided limits

We say that the limit of $f(x)$ as x approaches a from the left is L and write

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Similarly, we say that the limit of $f(x)$ as x approaches a from the right is L and write

$$\lim_{x \rightarrow a^+} f(x) = L,$$

if we can make $f(x)$ as close as we want to L by taking x close enough to a from the right (i.e., $x > a$).

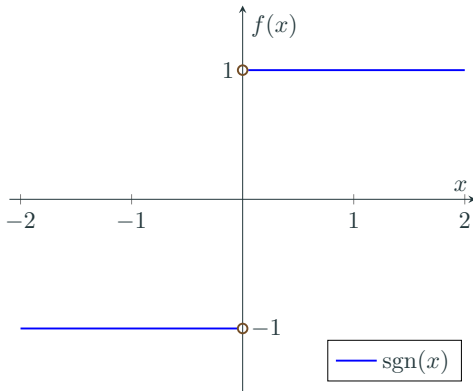
Limits

One-Sided Limits

Example

The signum function $\text{sgn}(x) = x/|x|$ has the following one-sided limits:

$$\lim_{x \rightarrow 0^-} \text{sgn}(x) = -1, \quad \lim_{x \rightarrow 0^+} \text{sgn}(x) = 1.$$



Limits

One-Sided Limits

Theorem. *Relationship between one-sided limits and two-sided limits*

Let f be a function defined on an open interval containing a , except possibly at a itself. Then:

- If $\lim_{x \rightarrow a^-} f(x) = L$ and $\lim_{x \rightarrow a^+} f(x) = L$, then $\lim_{x \rightarrow a} f(x) = L$.
- If either one of the one-sided limits does not exist or is not equal to L , then $\lim_{x \rightarrow a} f(x)$ does not exist.

Limits

One-Sided Limits

Example

If

$$f(x) = \frac{|x - 2|}{x^2 + x - 6},$$

find $\lim_{x \rightarrow 2^-} f(x)$ and $\lim_{x \rightarrow 2^+} f(x)$, and $\lim_{x \rightarrow 2} f(x)$.

Limits

One-Sided Limits

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Solution

$$\begin{aligned} f(x) &= \frac{|x - 2|}{(x - 2)(x + 3)} \\ &= \begin{cases} \frac{x-2}{(x-2)(x+3)} & \text{if } x > 2 \\ \frac{-(x-2)}{(x-2)(x+3)} & \text{if } x < 2 \end{cases} \\ &= \begin{cases} \frac{1}{x+3} & \text{if } x > 2 \\ -\frac{1}{x+3} & \text{if } x < 2 \end{cases} \end{aligned}$$

Limits

One-Sided Limits

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$$\begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= -\frac{1}{5}, \\ \lim_{x \rightarrow 2^+} f(x) &= \frac{1}{5}, \text{ and } \lim_{x \rightarrow 2} f(x) \\ &\text{does not exist.} \end{aligned}$$

Limits

One-Sided Limits

Example

What one-sided limits does $g(x) = \sqrt{1 - x^2}$ have at $x = 1$ and $x = -1$?

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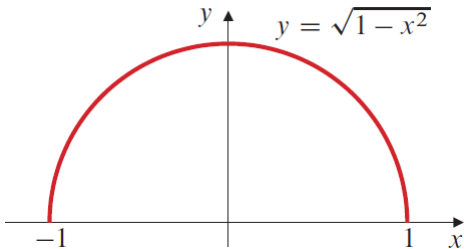
$$g(x) = \sqrt{1 - x^2}$$

$$\lim_{x \rightarrow 1^-} g(x) = 0,$$

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Limits

One-Sided Limits

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The one-sided limits at $x = 1$ are:

$$\lim_{x \rightarrow 1^-} g(x) = 0, \quad \lim_{x \rightarrow 1^+} g(x) = \text{undefined}.$$

The one-sided limits at $x = -1$ are:

$$\lim_{x \rightarrow -1^-} g(x) = \text{undefined}, \quad \lim_{x \rightarrow -1^+} g(x) = 0.$$

Limits

Rules for Calculating Limits

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If m is an integer and n is a positive integer, then:

- $\lim_{x \rightarrow a} f(x)^{m/n} = L^{m/n}$, provided $L^{m/n}$ makes sense as a real number (Power Rule)

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If $f(x) \leq g(x)$ on an interval containing a in its interior, then:

- $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$ (Order preservation)

Example

Evaluate

$$(a) \lim_{x \rightarrow -2} \frac{x^2 + x - 2}{x^2 + 5x + 6} \quad (b) \lim_{x \rightarrow a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} \quad (c) \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x^2 - 16}$$

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Solution

(a) We can factor the numerator and denominator:

$$\lim_{x \rightarrow -2} \frac{x^2 + x - 2}{x^2 + 5x + 6} = \lim_{x \rightarrow -2} \frac{(x - 1)(x + 2)}{(x + 2)(x + 3)}.$$

For $x \neq -2$, we can cancel $(x + 2)$:

$$= \lim_{x \rightarrow -2} \frac{x - 1}{x + 3} = \frac{-3}{1} = -3.$$

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Solution

(b) We can rewrite the expression:

$$\lim_{x \rightarrow a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} = \lim_{x \rightarrow a} \frac{a - x}{ax(x - a)} = \lim_{x \rightarrow a} \frac{-1}{ax} = -\frac{1}{a^2}.$$

Example

Evaluate

$$(a) \lim_{x \rightarrow -2} \frac{x^2 + x - 2}{x^2 + 5x + 6} \quad (b) \lim_{x \rightarrow a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} \quad (c) \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x^2 - 16}$$

Solution

(c) We can factor the denominator:

$$\begin{aligned} \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x^2 - 16} &= \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{\underbrace{(\sqrt{x} - 2)(\sqrt{x} + 2)}_{x-4}(x+4)} \\ &= \lim_{x \rightarrow 4} \frac{1}{(\sqrt{x} + 2)(x + 4)} = \frac{1}{32}. \end{aligned}$$

Theorem. *Limits of Polynomials and Rational Functions*

- If $f(x)$ is a polynomial, then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

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- If $g(x)$ and $h(x)$ are polynomials, then

$$\lim_{x \rightarrow a} \frac{g(x)}{h(x)} = \frac{g(a)}{h(a)},$$

provided $h(a) \neq 0$.

Example

Find

$$(a) \lim_{x \rightarrow 0} \frac{x^2 + x + 4}{x^3 - 2x^2 + 7},$$

$$(b) \lim_{x \rightarrow 2} \sqrt{2x + 1}.$$

Solution

Example

Find

$$(a) \lim_{x \rightarrow 0} \frac{x^2 + x + 4}{x^3 - 2x^2 + 7},$$

$$(b) \lim_{x \rightarrow 2} \sqrt{2x + 1}.$$

Solution

(a) We can substitute $x = 0$:

$$\lim_{x \rightarrow 0} \frac{x^2 + x + 4}{x^3 - 2x^2 + 7} = \frac{0^2 + 0 + 4}{0^3 - 2(0)^2 + 7} = \frac{4}{7}.$$

Example

Find

$$(a) \lim_{x \rightarrow 0} \frac{x^2 + x + 4}{x^3 - 2x^2 + 7},$$

$$(b) \lim_{x \rightarrow 2} \sqrt{2x + 1}.$$

Solution

(b) We can substitute $x = 2$:

$$\lim_{x \rightarrow 2} \sqrt{2x + 1} = \sqrt{\lim_{x \rightarrow 2} (2x + 1)} = \sqrt{2 \cdot 2 + 1} = \sqrt{5}.$$

Example

Find the following limits:

$$(a) \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h},$$

$$(b) \lim_{t \rightarrow 0} \frac{1}{t\sqrt{1+t}} - \frac{1}{t}.$$

Solution

Example

Find the following limits:

$$(a) \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h},$$

$$(b) \lim_{t \rightarrow 0} \frac{1}{t\sqrt{1+t}} - \frac{1}{t}.$$

Solution

(a) We can multiply the numerator and denominator by the conjugate:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h} &= \lim_{h \rightarrow 0} \frac{(\sqrt{1+h} - 1)(\sqrt{1+h} + 1)}{h(\sqrt{1+h} + 1)} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{1+h} + 1)} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{2}. \end{aligned}$$

Limits

Example

Find the following limits:

$$(a) \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h},$$

$$(b) \lim_{t \rightarrow 0} \frac{1}{t\sqrt{1+t}} - \frac{1}{t}.$$

Solution

(b) We can combine the fractions:

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t\sqrt{1+t}} - \frac{1}{t} &= \lim_{t \rightarrow 0} \frac{1 - \sqrt{1+t}}{t\sqrt{1+t}} = \lim_{t \rightarrow 0} \frac{(1 - \sqrt{1+t})(1 + \sqrt{1+t})}{t\sqrt{1+t}(1 + \sqrt{1+t})} \\ &= \lim_{t \rightarrow 0} \frac{-t}{t\sqrt{1+t}(1 + \sqrt{1+t})} \\ &= \lim_{t \rightarrow 0} \frac{-1}{\sqrt{1+t}(1 + \sqrt{1+t})} = -\frac{1}{2}. \end{aligned}$$

Limits

The Squeeze Theorem

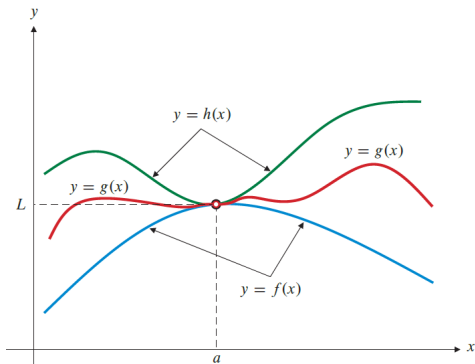
Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ for all x in some open interval containing a , except possibly at a itself, and

$$\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x),$$

then

$$\lim_{x \rightarrow a} g(x) = L.$$



Limits

The Squeeze Theorem

Example

Given that $3 - x^2 \leq u(x) \leq 3 + x^2$ for all $x \neq 0$, find $\lim_{x \rightarrow 0} u(x)$.

Limits

The Squeeze Theorem

Example

Given that $3 - x^2 \leq u(x) \leq 3 + x^2$ for all $x \neq 0$, find $\lim_{x \rightarrow 0} u(x)$.

Solution

Since $3 - x^2$ and $3 + x^2$ both approach 3 as x approaches 0, we can apply the Squeeze Theorem:

$$\lim_{x \rightarrow 0} u(x) = 3.$$

Limits

The Squeeze Theorem

Example

Show that if $\lim_{x \rightarrow a} |f(x)| = 0$, then $\lim_{x \rightarrow a} f(x) = 0$.

Limits

The Squeeze Theorem

Example

Show that if $\lim_{x \rightarrow a} |f(x)| = 0$, then $\lim_{x \rightarrow a} f(x) = 0$.

Solution

Since $|f(x)| \geq 0$ for all x , we can apply the Squeeze Theorem:

$$-|f(x)| \leq f(x) \leq |f(x)|.$$

As x approaches a , both $-|f(x)|$ and $|f(x)|$ approach 0. Therefore, by the Squeeze Theorem:

$$\lim_{x \rightarrow a} f(x) = 0.$$

Limits

The Squeeze Theorem

Example

Show that $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Solution

Since $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for all $x \neq 0$, we can apply the Squeeze Theorem:

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

As x approaches 0, both $-x^2$ and x^2 approach 0. Therefore, by the Squeeze Theorem:

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Limits

The Squeeze Theorem

Example

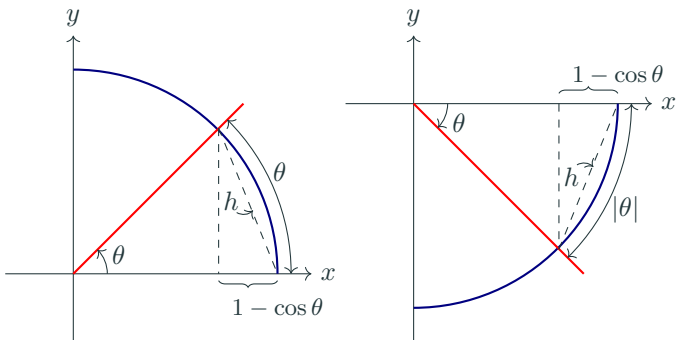
We show that $\lim_{\theta \rightarrow 0} \cos \theta = 1$ and $\lim_{\theta \rightarrow 0} \sin \theta = 0$ using the Squeeze Theorem.

Limits

The Squeeze Theorem

Example

We show that $\lim_{\theta \rightarrow 0} \cos \theta = 1$ and $\lim_{\theta \rightarrow 0} \sin \theta = 0$ using the Squeeze Theorem.



$$0 \leq 1 - \cos \theta \leq h \leq |\theta| \implies \lim_{\theta \rightarrow 0} (1 - \cos \theta) = 0 \implies \lim_{\theta \rightarrow 0} \cos \theta = 1$$

Limits

The Squeeze Theorem

Example

We show that $\lim_{\theta \rightarrow 0} \cos \theta = 1$ and $\lim_{\theta \rightarrow 0} \sin \theta = 0$ using the Squeeze Theorem.

