MAT123 MATHEMATICS I

Lecture 06: Limits and Continuity

Outline

Limits at Infinity and Infinite Limits

Limits at Infinity

- Limits at infinity describe the behavior of a function as the input grows larger and larger without bound.

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- If $f(x)$ approaches a finite value L, we say the limit exists and is equal to L as x approaches infinity.
- If $f(x)$ increases without bound, we write $\lim_{x \to \infty} f(x) = \infty$.

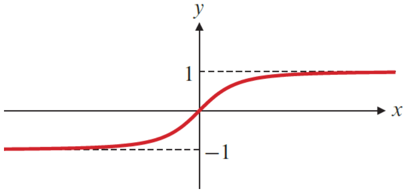
Limits at Infinity

Consider the function

$$\frac{x}{\sqrt{x^2+1}}.$$

We can make the following table of values and plot the function to understand its behavior as x approaches infinity.

x	$\frac{x}{\sqrt{x^2+1}}$
-1000	-0.9999
-100	-0.999
-10	-0.995
-1	-0.707
0	0.000
1	0.707
10	0.995
100	0.999
1000	0.9999



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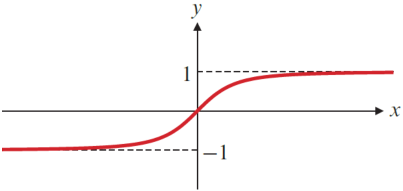

As x approaches infinity, the function $\frac{x}{\sqrt{x^2+1}}$ approaches 1. Thus, we can conclude that:

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1.$$

Informal Definition of Limits at Infinity

- If the function f is defined on (a, ∞) and we can make sure that the values of $f(x)$ are arbitrarily close to L by taking x large enough, we say that $\lim_{x \to \infty} f(x) = L$.

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- If the function f is defined on $(-\infty, a)$ and we can make sure that the values of $f(x)$ are arbitrarily close to L by taking x small enough, we say that $\lim_{x\to-\infty} f(x) = L$.

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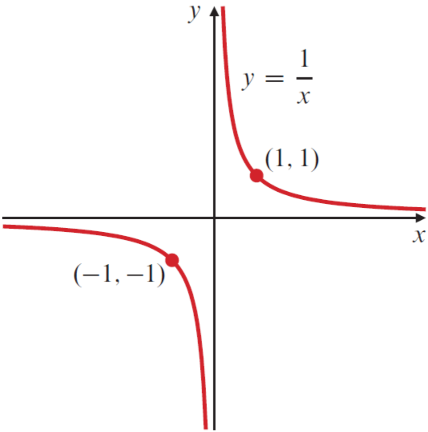
Limits at Infinity: Example

Consider the function $f(x) = \frac{1}{x}$.

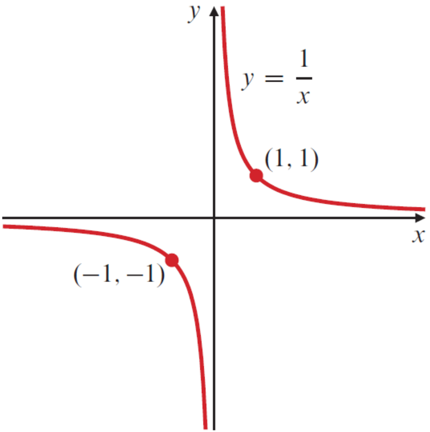


As x approaches infinity, the function approaches 0. Thus, we can conclude that:

$$\lim_{x \to \infty} f(x) = 0.$$

Limits at Infinity: Example

Consider the function $f(x) = \frac{1}{x}$.



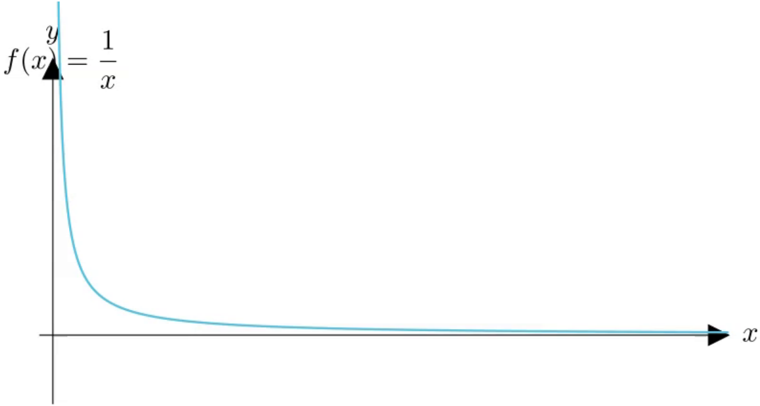
As x approaches infinity, the function approaches 0. Thus, we can conclude that:

$$\lim_{x \to \infty} f(x) = 0.$$

As x approaches negative infinity, the function also approaches 0. Thus, we can conclude that:

$$\lim_{x \to -\infty} f(x) = 0.$$

Limits at Infinity: Example



Limits at Infinity: Example

Example

Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

Solution

Limits at Infinity: Example

Example

Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \frac{x}{\sqrt{x^2+1}}$.

Solution

- *As x approaches infinity, we can divide the numerator and denominator by x:*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1.$$

Thus, we conclude that:

$$\lim_{x\to\infty} f(x) = 1.$$

Example

Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

Solution

- *As x approaches negative infinity, we can again divide the numerator and denominator by x:*

$$\begin{aligned}\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} \\ &= \lim_{x\to-\infty} \frac{x}{-x\sqrt{1+\frac{1}{x^2}}} = -1.\end{aligned}$$

Thus, we conclude that:

$$\lim_{x\to-\infty} f(x) = -1.$$

Limits at Infinity: Example

Example

Evaluate $\lim\limits_{x \to \pm\infty} \dfrac{2x^2 - x + 3}{3x^2 + 5}$.

Solution

Limits at Infinity: Example

Example

Evaluate $\lim_{x \to \pm\infty} \frac{2x^2 - x + 3}{3x^2 + 5}$.

Solution

- *As x approaches infinity, we can divide the numerator and denominator by x^2:*

$$\lim_{x \to \infty} \frac{2x^2 - x + 3}{3x^2 + 5} = \lim_{x \to \infty} \frac{2 - \frac{1}{x} + \frac{3}{x^2}}{3 + \frac{5}{x^2}} = \frac{2}{3}.$$

Thus, we conclude that:

$$\lim_{x \to \infty} f(x) = \frac{2}{3}.$$

Limits at Infinity: Example

Example

Evaluate $\lim_{x \to \pm\infty} \frac{2x^2 - x + 3}{3x^2 + 5}$.

Solution

- *As x approaches negative infinity, we can again divide the numerator and denominator by x^2:*

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{2x^2 - x + 3}{3x^2 + 5}$$
$$= \lim_{x \to -\infty} \frac{2 - \frac{1}{x} + \frac{3}{x^2}}{3 + \frac{5}{x^2}} = \frac{2}{3}.$$

Thus, we conclude that:

$$\lim_{x \to -\infty} f(x) = \frac{2}{3}.$$

Limits at Infinity: Example

Example

Evaluate $\lim_{x \to \pm\infty} \frac{5x + 2}{2x^3 - 1}$.

Limits at Infinity: Example

Example

Evaluate $\lim_{x\to\pm\infty} \dfrac{5x+2}{2x^3-1}$.

Solution

As x approaches infinity or negative infinity, we can divide the numerator and denominator by x^3:

$$\lim_{x\to\pm\infty} \frac{5x+2}{2x^3-1} = \lim_{x\to\pm\infty} \frac{\frac{5}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}} = 0.$$

Thus, we conclude that:

$$\lim_{x\to\pm\infty} f(x) = 0.$$

Limits at Infinity: Rational Functions

Consider the rational function $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. The limit of $f(x)$ as x approaches infinity or negative infinity can be determined by comparing the degrees of the polynomials.

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$$\lim_{x \to \pm\infty} f(x) = 0.$$

- If the degree of $p(x)$ is equal to the degree of $q(x)$, then:

$$\lim_{x \to \pm\infty} f(x) = \frac{a}{b},$$

  where a and b are the leading coefficients of $p(x)$ and $q(x)$, respectively.

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  where a and b are the leading coefficients of $p(x)$ and $q(x)$, respectively.

- If the degree of $p(x)$ is greater than the degree of $q(x)$, then:

$$\lim_{x \to \pm\infty} f(x) = \infty \text{ or } -\infty.$$

Limits at Infinity: Rational Functions Example

Example

Evaluate $\lim_{x \to \infty} \left(\sqrt{x^2 + x} - x \right)$.

Limits at Infinity: Rational Functions Example

Example

Evaluate $\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right)$.

Solution

To evaluate the limit, we can multiply and divide by the conjugate:

$$\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right) = \lim_{x\to\infty} \frac{(\sqrt{x^2+x} - x)(\sqrt{x^2+x} + x)}{\sqrt{x^2+x} + x}.$$

Limits at Infinity: Rational Functions Example

Example

Evaluate $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$.

Solution

Therefore, we conclude that:

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\frac{1}{2}.$$

Limits at Infinity: arctan

Example

Evaluate $\lim_{x \to \infty} \arctan(x)$.

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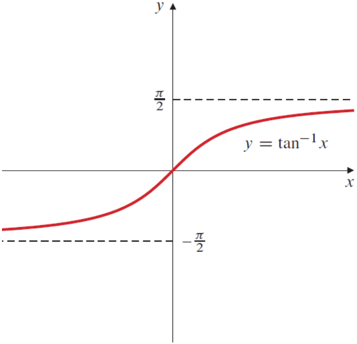
Solution



As x approaches infinity, the arctangent function approaches $\frac{\pi}{2}$:

$$\lim_{x \to \infty} \arctan(x) = \frac{\pi}{2}.$$

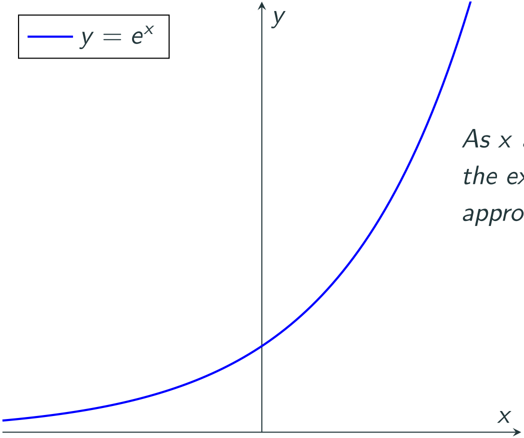
As x approaches negative infinity, the arctangent function approaches $-\frac{\pi}{2}$:

$$\lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}.$$

Example

Evaluate $\lim_{x\to\infty} e^x$.

Solution



As x approaches negative infinity, the exponential function e^x approaches 0:

$$\lim_{x\to-\infty} e^x = 0.$$

Infinite Limits

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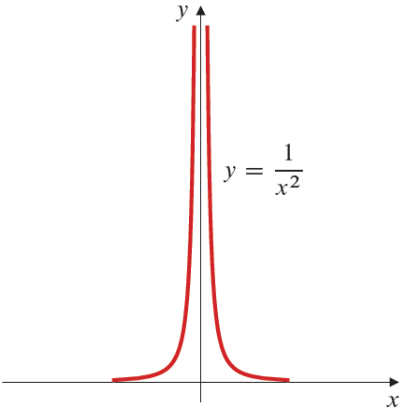
- Infinite limits often occur in rational functions where the denominator approaches zero while the numerator approaches a non-zero value.

Infinite Limits: Example

Example

Describe the behaviour of the function $\frac{1}{x^2}$ near $x = 0$.

Solution



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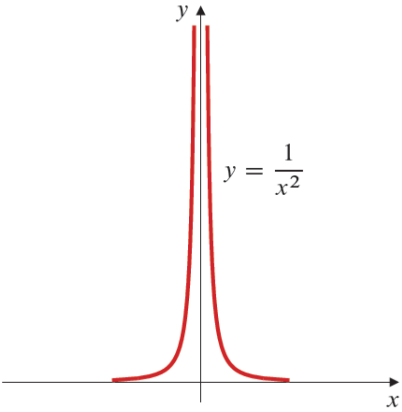

As x approaches 0 from the right ($x \to 0^+$), the function $\frac{1}{x^2}$ increases without bound:

$$\lim_{x \to 0^+} \frac{1}{x^2} = \infty.$$

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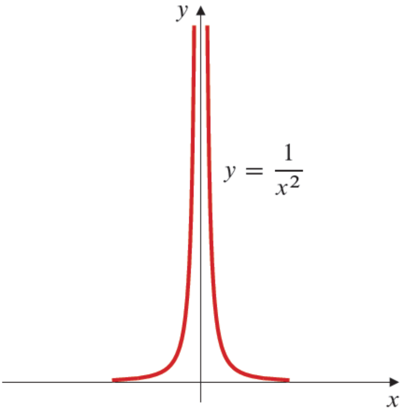


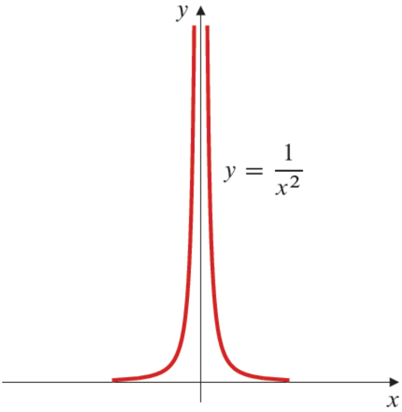
As x approaches 0 from the left ($x \to 0^-$), the function also increases without bound:

$$\lim_{x \to 0^-} \frac{1}{x^2} = \infty.$$

Example

Describe the behaviour of the function $\frac{1}{x^2}$ near $x = 0$.

Solution



Therefore, we can conclude that:

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Informal Definition of Infinite Limits

- If the function f is defined on an interval around a (except possibly at a) and we can make sure that the values of $f(x)$ are arbitrarily large (positive or negative) by taking x close enough to a, we say that $f(x)$ approaches ∞ or $-\infty$, according to whether the values of $f(x)$ are positive (or negative) for all x sufficiently close to a.

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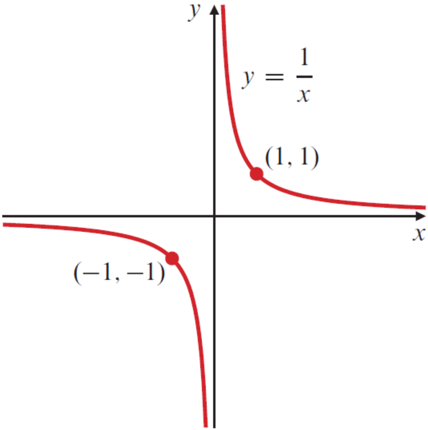
- We can express this mathematically as:

$$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad \lim_{x \to a} f(x) = -\infty.$$

Example

Describe the behaviour of the function $\frac{1}{x}$ near $x = 0$.

Solution



Infinite Limits: Example

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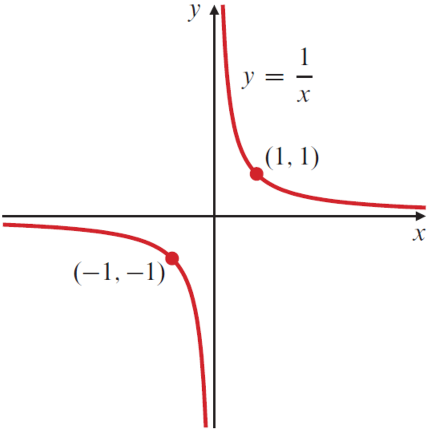


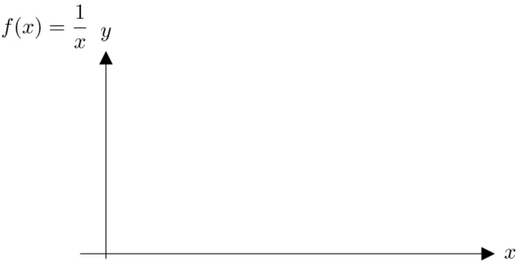
As x approaches 0 from the right ($x \to 0^+$), the function $\frac{1}{x}$ increases without bound:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$

Example

Describe the behaviour of the function $\dfrac{1}{x}$ near $x = 0$.

Solution



Infinite Limits: Example

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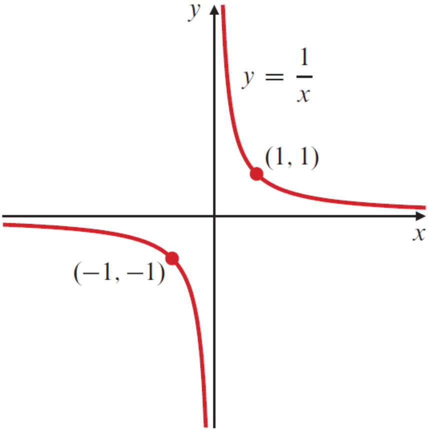


As x approaches 0 from the left ($x \to 0^-$), the function decreases without bound:

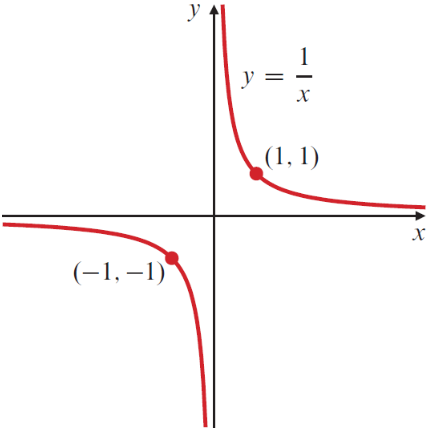
$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Infinite Limits: Example

Example

Describe the behaviour of the function $\dfrac{1}{x}$ near $x = 0$.

Solution



Therefore, we can conclude that:

$$\lim_{x \to 0} f(x)$$

does not exist, as the left-hand limit and right-hand limit do not match.

Infinite Limits: Example

Example (Polynomial behaviour at infinity)

We have the following limits:

(a) $\lim_{x \to \infty} (3x^3 - x^2 + 2) = \infty$ **(b)** $\lim_{x \to -\infty} (3x^3 - x^2 + 2) = -\infty$

(c) $\lim_{x \to \infty} (x^4 - 5x^3 - x) = \infty$ **(d)** $\lim_{x \to -\infty} (x^4 - 5x^3 - x) = \infty$

Infinite Limits: Example

Example (Rational functions with numerator of higher degree)

Evaluate the limit $\lim_{x \to \infty} \frac{x^3 + 1}{x^2 + 1}$.

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Evaluate the limit $\lim_{x \to \infty} \frac{x^3 + 1}{x^2 + 1}$.

Solution

To evaluate the limit, we can divide the numerator and denominator by x^3:

$$\lim_{x \to \infty} \frac{x^3 + 1}{x^2 + 1} = \lim_{x \to \infty} \frac{1 + \frac{1}{x^3}}{\frac{1}{x} + \frac{1}{x^3}} = \infty.$$

Example (Rational functions, cont.)

(a) $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = \frac{0}{4} = 0$

Infinite Limits: Example

Example (Rational functions, cont.)

(a) $\lim_{x \to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x \to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x \to 2} \frac{x-2}{x+2} = \frac{0}{4} = 0$

(b) $\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4} = 0.25$

Infinite Limits: Example

Example (Rational functions, cont.)

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(c) $\lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$

Infinite Limits: Example

Example (Rational functions, cont.)

(a) $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = \frac{0}{4} = 0$

(b) $\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4} = 0.25$

(c) $\lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$

(d) $\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$

Infinite Limits: Example

Example (Rational functions, cont.)

(a) $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = \frac{0}{4} = 0$

(b) $\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4} = 0.25$

(c) $\lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$

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(e) $\lim_{x\to 2} \frac{x-3}{x^2-4}$ does not exist.

Infinite Limits: Example

Example (Rational functions, cont.)

(a) $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = \frac{0}{4} = 0$

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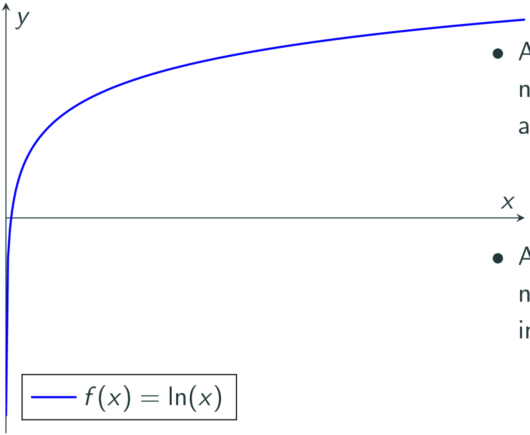
(d) $\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$

(e) $\lim_{x\to 2} \frac{x-3}{x^2-4}$ does not exist.

(f) $\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$

Infinite Limits: Example

Consider the function $f(x) = \ln(x)$.



- As x approaches 0^+, the natural logarithm function approaches $-\infty$:

$$\lim_{x \to 0^+} f(x) = -\infty.$$

- As x approaches infinity, the natural logarithm function increases without bound:

$$\lim_{x \to \infty} f(x) = \infty.$$

Continuity

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- Recall that if $f(x)$ is a polynomial function, then its limit at a is calculated as:

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- Such a limit property is referred to as **continuity at a point**.

Continuity

Definition *(Interior Point)*

An interior point of a set S is a point c such that there exists an interval $(c-\delta, c+\delta)$ contained in S for some $\delta > 0$.

Remark.

The points of an interval are classified as either interior points or end points.

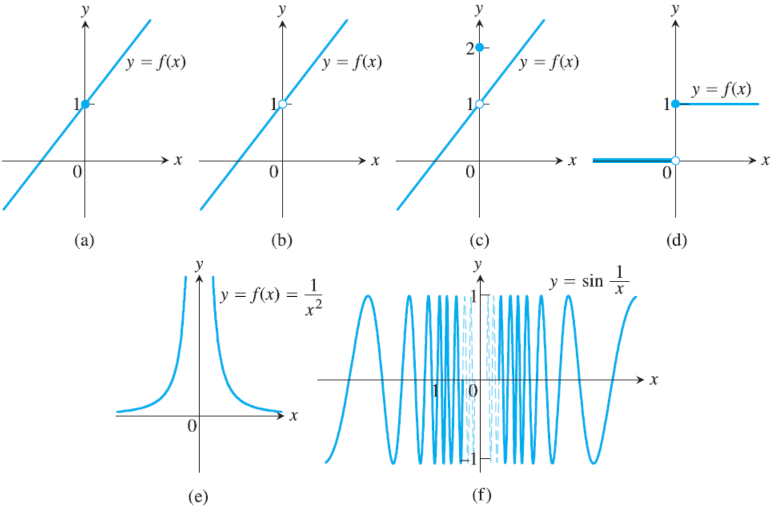
Definition *(Continuity at a Point)*

Suppose that the domain of a function $f(x)$ is either an interval or a union of intervals.
We say that $f(x)$ is **continuous** at an interior point c in the domain if

$$\lim_{x \to c} f(x) = f(c).$$

If either the limit does not exist or is not equal to $f(c)$, we say that $f(x)$ is **discontinuous** at c.

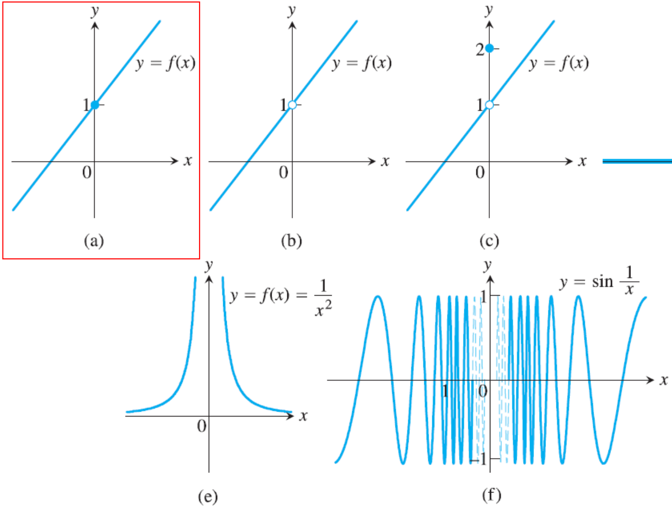
Continuity



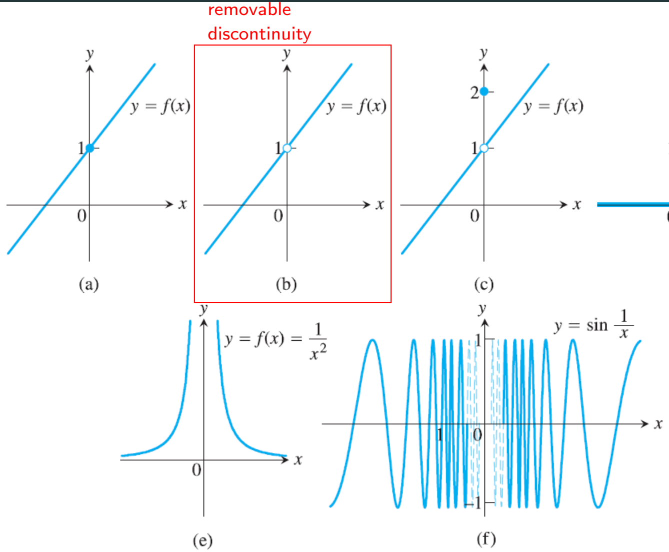
y
y = f(x)
1
0
x
(a)
y
y = f(x)
1
0
x
(b)
y
2
y = f(x)
1
0
x
(c)
y
y = f(x)
1
0
x
(d)
y
$y = f(x) = \frac{1}{x^2}$
0
x
(e)
y
$y = \sin \frac{1}{x}$
1
−1
0
x
−1
(f)

Continuity

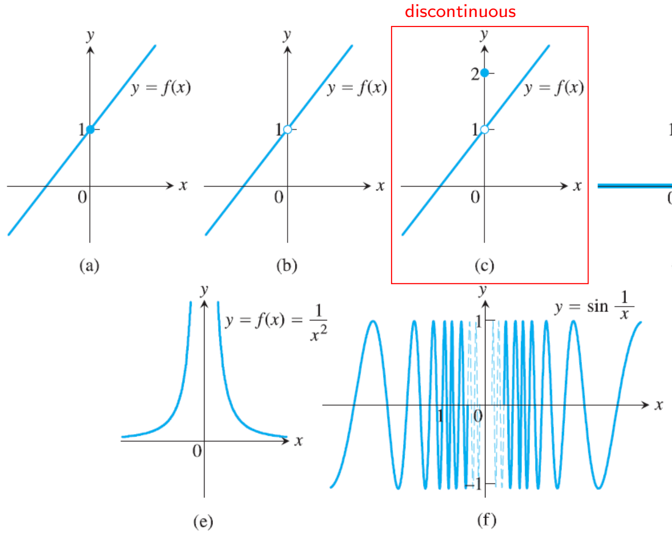
continuous



Continuity


removable discontinuity
y = f(x)
(a)
(b)
(c)
(d)
$y = f(x) = \frac{1}{x^2}$
(e)
$y = \sin \frac{1}{x}$
(f)

Continuity



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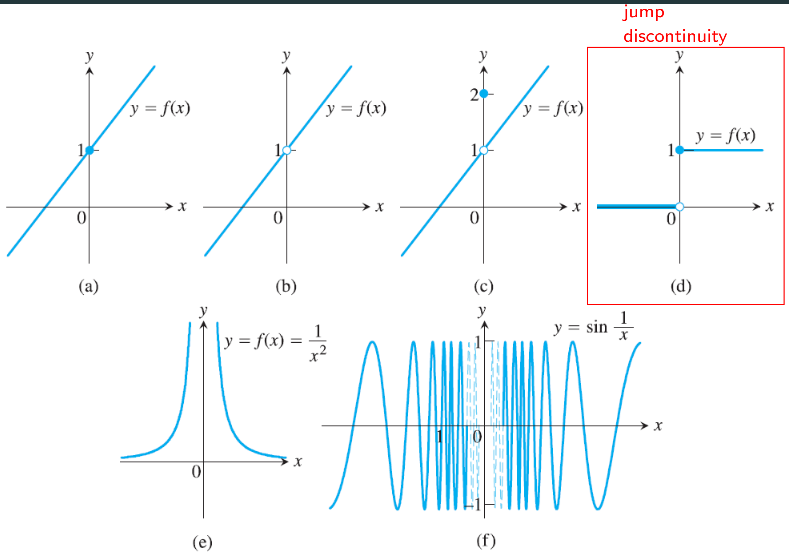
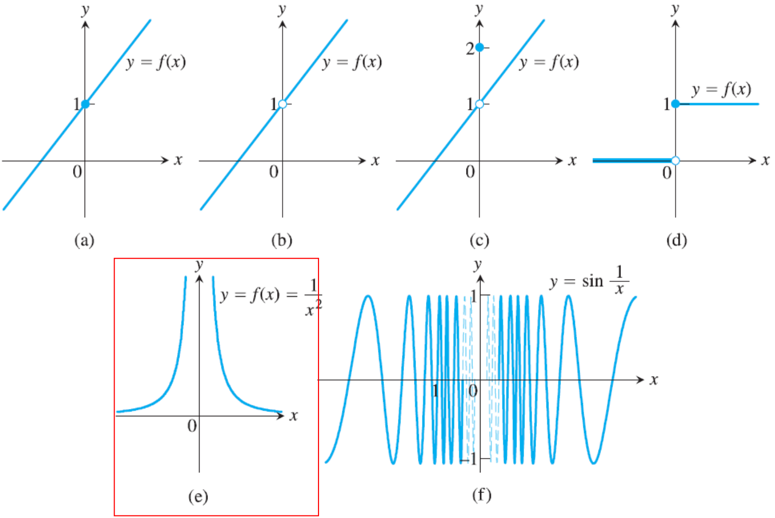
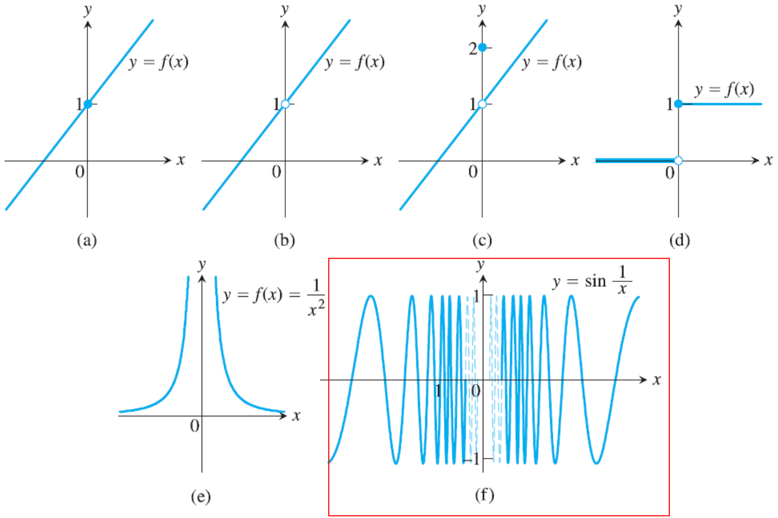


Continuity



infinite
discontinuity

Continuity



oscillating
discontinuity

A function $f(x)$ is continuous at an interior point c of its domain if and only if it meets the following three conditions:

1. $f(c)$ is defined (i.e., c is in the domain of f).
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

Definition *(Right Continuity)*

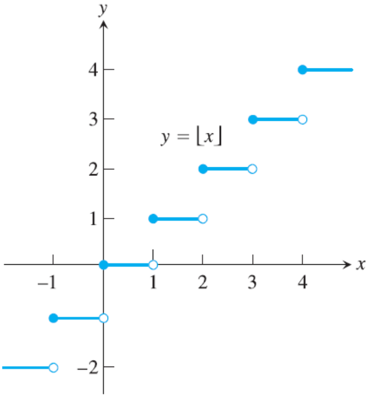
A function $f(x)$ is said to be **right continuous** at a point c if

$$\lim_{x \to c^+} f(x) = f(c).$$

Definition *(Left Continuity)*

A function $f(x)$ is said to be **left continuous** at a point c if

$$\lim_{x \to c^-} f(x) = f(c).$$



The greatest integer function is continuous at every noninteger point. It is right continuous, but not left continuous, at every integer point.

Theorem.

Function f is continuous at a point c if and only if it is both right continuous and left continuous at that point.

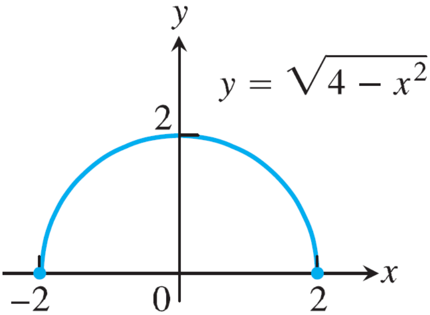
Definition.

A function $f(x)$ is said to be **continuous** at a left endpoint a of its domain if

$$\lim_{x \to a^+} f(x) = f(a).$$

A function $f(x)$ is said to be **continuous** at a right endpoint b if

$$\lim_{x \to b^-} f(x) = f(b).$$



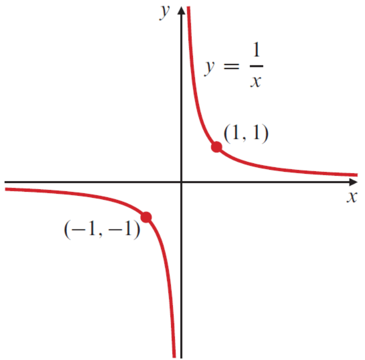
$y = \sqrt{4 - x^2}$ is continuous at every point in its domain, which is the interval $[-2, 2]$.

$$\lim_{x \to 2^-} = 0 = f(2)$$

$$\lim_{x \to -2^+} = 0 = f(-2)$$

Definition.

A function $f(x)$ is said to be **continuous on an interval** I if it is continuous at every point in I.
In particular, we will say that f is a **continuous function** if it is continuous on its entire domain.



$f(x) = \frac{1}{x}$ is continuous on its entire domain, which is $(-\infty, 0) \cup (0, \infty)$.

There are many continuous functions.

- all polynomial functions;
- all rational functions;
- the sine, cosine, tangent, cotangent, secant, and cosecant functions; and
- all rational powers $x^{p/q}$, where p and q are integers and $q \neq 0$;
- the absolute value function $f(x) = |x|$.

Theorem *(Combining Continuous Functions)*

If f and g are continuous at a point c, then the following functions are also continuous at c:

- the sum $f + g$;
- the difference $f - g$;
- the product $f \cdot g$;
- the quotient $\frac{f}{g}$ (if $g(c) \neq 0$);
- the nth root $\sqrt[n]{f}$ (if $f(c) \geq 0$ for even n).

Theorem *(Composition of Continuous Functions)*

If $g(f(x))$ is defined on an interval containing c, and if g is continuous at L, where

$$\lim_{x \to c} f(x) = L,$$

then

$$\lim_{x \to c} g(f(x)) = g(L) = g(\lim_{x \to c} f(x))$$

In particular, if f is continuous at c, then the composition $g \circ f$ is continuous at c.

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- $f(x) = \sqrt{x^2 - 2x - 5}$ (domain: $[0, 2 + \sqrt{9}]$)
- $f(x) = \frac{|x|}{\sqrt{|x+2|}}$ (domain: $\mathbb{R} \setminus \{-2\}$)

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Then we can define a new function $F(x)$ by

$$F(x) = \begin{cases} f(x) & \text{if } x \neq c \\ L & \text{if } x = c \end{cases}$$

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$$F(x) = \begin{cases} f(x) & \text{if } x \neq c \\ L & \text{if } x = c \end{cases}$$

Note that the function $F(x)$ is continuous at c. It is called the **continuous extension** of $f(x)$ at c. We also say that $f(x)$ has a **removable discontinuity** at c.

Example

Show that $f(x) = \dfrac{x^2 - x}{x^2 - 1}$ has a removable discontinuity at $x = 1$, and find its continuous extension.

Solution

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- The function $f(x)$ is not defined at $x = 1$ because the denominator becomes zero.
- However, it is equal to the simplified function for $x \neq 1$:

$$f(x) = \frac{x^2 - x}{x^2 - 1} = \frac{x(x-1)}{(x-1)(x+1)} = \frac{x}{x+1} \quad (x \neq 1).$$

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- The limit as x approaches 1 is:

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2}.$$

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Solution

- Thus, we can define the continuous extension $F(x)$ as:

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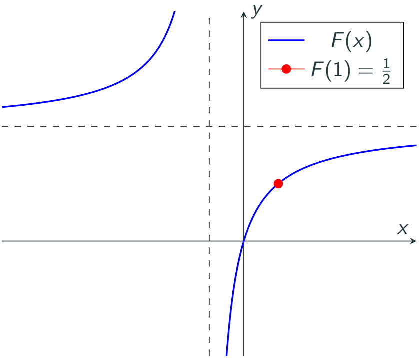
- 

Theorem.

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

Continuity

Continuous Extensions and Removable Discontinuities

Theorem.

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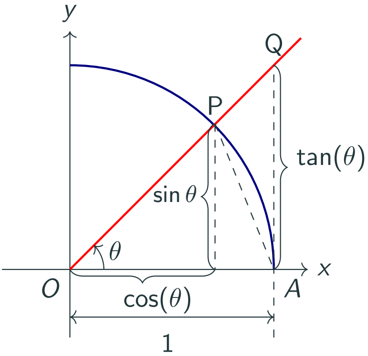


$$\text{area}(\overset{\triangle}{AOP}) \le \text{area}(\overset{\frown}{AOP}) \le \text{area}(\overset{\triangle}{AOQ})$$

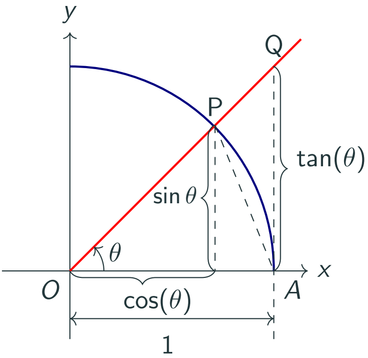
$$\frac{\sin\theta}{2} \le \frac{\theta}{2} \le \frac{\tan\theta}{2}$$

Dividing by $\frac{\sin\theta}{2}$ gives:

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}.$$

Theorem.

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$



By the Squeeze Theorem, we have:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

Example

Show that $f(x) = \dfrac{\sin(x)}{x}$ has a removable discontinuity at $x = 0$, and find its continuous extension.

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Solution

- *The function $f(x)$ is not defined at $x = 0$ because the denominator becomes zero.*
- *However, we can find the limit as x approaches 0:*

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

- *Thus, we can define a new function $F(x)$ by*

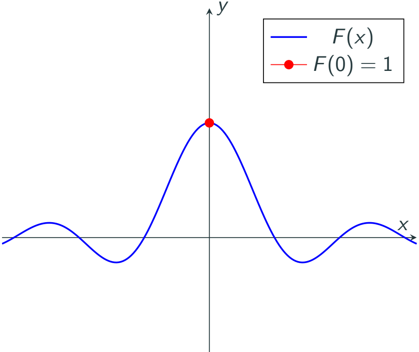
$$F(x) = \begin{cases} f(x) & \textit{if } x \neq 0 \\ 1 & \textit{if } x = 0 \end{cases}$$

Example

Show that $f(x) = \dfrac{\sin(x)}{x}$ has a removable discontinuity at $x = 0$, and find its continuous extension.

Solution

The function $F(x)$ is continuous at $x = 0$, and we say that $f(x)$ has a removable discontinuity at $x = 0$.



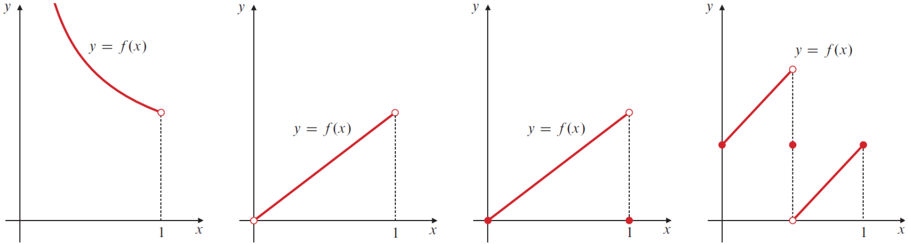
$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Theorem *(The Extreme Value Theorem)*

If f is continuous on a closed, finite interval $[a, b]$, then f attains both a maximum and a minimum value on that interval.
More precisely, there exist points c and d in $[a, b]$ such that:

$$f(c) \leq f(x) \leq f(d)$$

for all x in $[a, b]$.



y
y = f(x)
1
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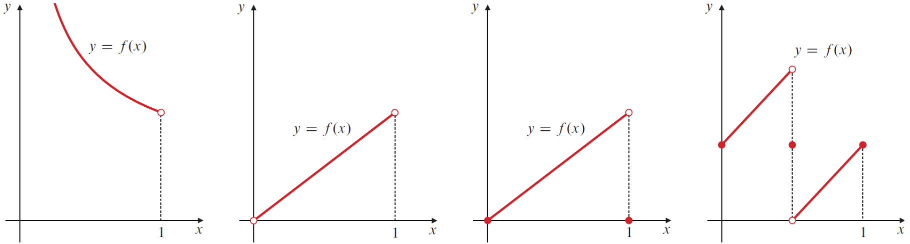
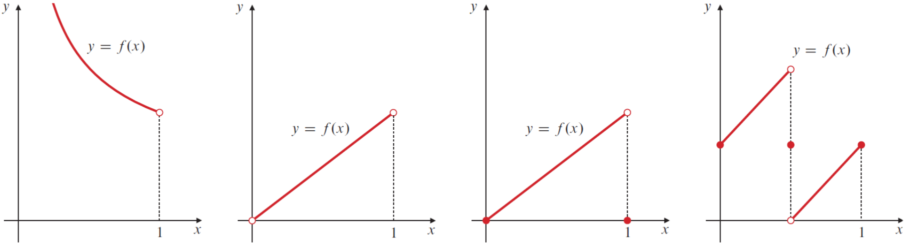


f is continuous on $(0, 1)$, but has neither maximum nor minimum on the interval $(0, 1)$



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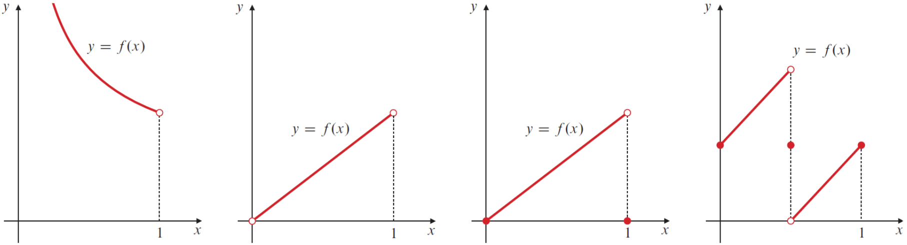
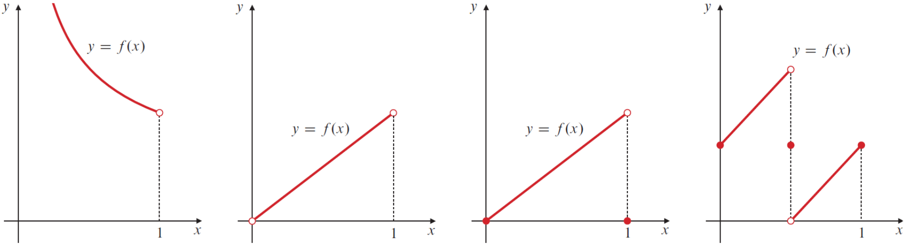


f is defined on $[0, 1]$, has maximum, but not minimum.



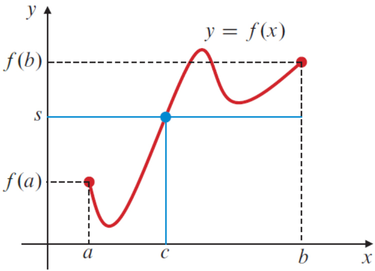
f is defined on $[0, 1]$, but has neither minimum, nor maximum.

Theorem.

If f is continuous on a closed, finite interval $[a, b]$, and if s is any number between $f(a)$ and $f(b)$, then there exists at least one point c in (a, b) such that

$$f(c) = s.$$

In other words, the function f takes on every value between $f(a)$ and $f(b)$ at least once in the interval (a, b).



Determining signs of a continuous function

Between two consecutive roots of a continuous function, the function must take on either all positive or all negative values, but not both.

More precisely, if f is continuous on an interval containing c and d such that $f(c) = f(d) = 0$, but $f(x) \neq 0$ for all x in the open interval (c, d), then either $f(x) > 0$ or $f(x) < 0$ for all x in (c, d).

Example

Determine the intervals on which the function $f(x) = x^3 - 4x$ is positive and negative.

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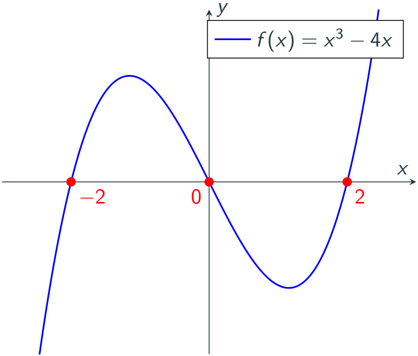
Solution



		-2		0		2	
x	$-$		$-$	$\circ$	$+$		$+$
$x^2 - 4$	$+$	$\circ$	$-$		$-$	$\circ$	$+$
$x^3 - 4x$	$-$		$+$		$-$		$+$

Example

Show that the equation $x^3 - x - 1 = 0$ has a root in the interval $[1, 2]$.

Example

Show that the equation $x^3 - x - 1 = 0$ has a root in the interval $[1, 2]$.

Solution

We can apply the Intermediate Value Theorem. First, we evaluate the function at the endpoints of the interval:

$$f(1) = 1^3 - 1 - 1 = -1,$$

$$f(2) = 2^3 - 2 - 1 = 5.$$

Since $f(1) < 0$ *and* $f(2) > 0$, *and* f *is continuous on* $[1, 2]$, *there must be at least one root* c *in* $(1, 2)$ *such that* $f(c) = 0$.

Example (The Bisection Method)

Solve the equation $x^3 - x - 1 = 0$ using the bisection method with an initial interval $[1, 2]$ correct to three decimal places.

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Solve the equation $x^3 - x - 1 = 0$ using the bisection method with an initial interval $[1, 2]$ correct to three decimal places.

Solution

Bisection Number	x	$f(x)$	Root in Interval	Midpoint
	1	−1		
	2	5	[1, 2]	1.5
1	1.5	0.8750	[1, 1.5]	1.25
2	1.25	−0.2969	[1.25, 1.5]	1.375
3	1.375	0.2246	[1.25, 1.375]	1.3125
4	1.3125	−0.0515	[1.3125, 1.375]	1.3438
5	1.3438	0.0826	[1.3125, 1.3438]	1.3282
6	1.3282	0.0147	[1.3125, 1.3282]	1.3204
7	1.3204	−0.0186	[1.3204, 1.3282]	1.3243
8	1.3243	−0.0018	[1.3243, 1.3282]	1.3263
9	1.3263	0.0065	[1.3243, 1.3263]	1.3253
10	1.3253	0.0025	[1.3243, 1.3253]	1.3248
11	1.3248	0.0003	[1.3243, 1.3248]	1.3246
12	1.3246	−0.0007	[1.3246, 1.3248]	

The root is 1.325, rounded to 3 decimal places.