

MAT123 MATHEMATICS I

Lecture 07: Limits and Continuity, Differentiation

Outline

Continuous Functions on Closed, Finite Intervals

The Intermediate Value Theorem

The Precise Definition of Limit

One-Sided Limits

Limits at Infinity

Infinite Limits

Conclusion

Differentiation

 Tangent Lines to Curves

Continuous Functions on Closed, Finite Intervals

Continuity

Continuous Functions on Closed, Finite Intervals

Theorem (*The Extreme Value Theorem*)

If f is continuous on a closed, finite interval $[a, b]$, then f attains both a maximum and a minimum value on that interval.

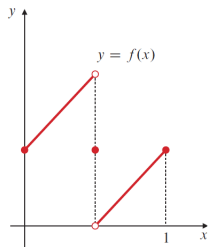
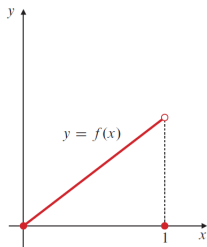
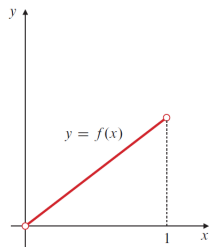
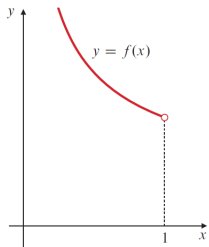
More precisely, there exist points c and d in $[a, b]$ such that:

$$f(c) \leq f(x) \leq f(d)$$

for all x in $[a, b]$.

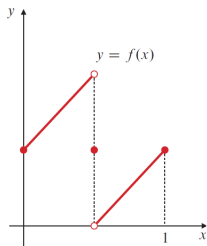
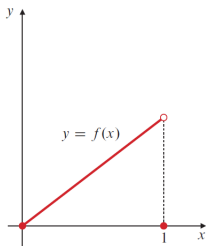
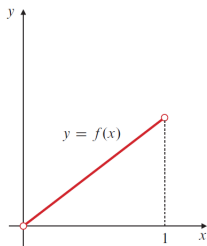
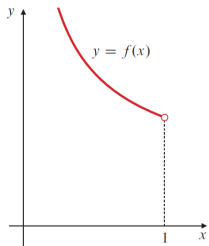
Continuity

Continuous Functions on Closed, Finite Intervals



Continuity

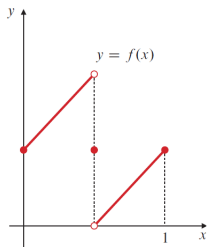
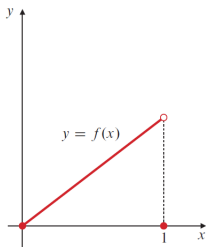
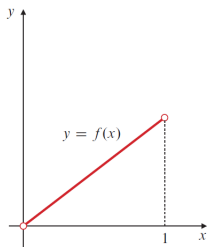
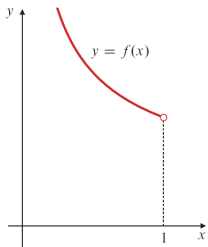
Continuous Functions on Closed, Finite Intervals



f is continuous on $(0,1)$, but has neither maximum nor minimum on the interval $(0,1)$

Continuity

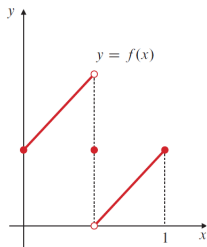
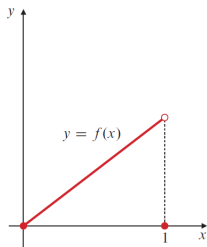
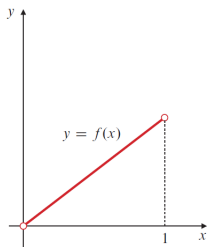
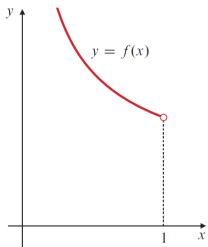
Continuous Functions on Closed, Finite Intervals



f has neither maximum nor minimum on the interval $(0, 1)$

Continuity

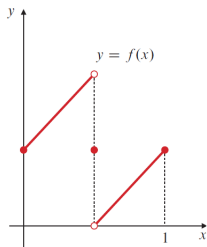
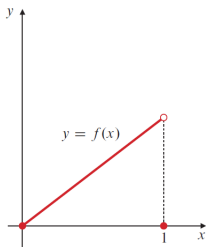
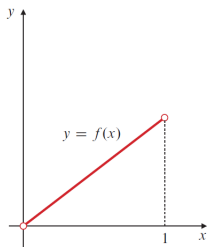
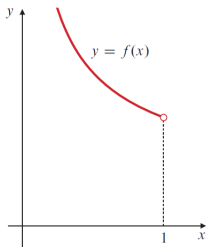
Continuous Functions on Closed, Finite Intervals



f is defined on $[0, 1]$,
has maximum, but
not minimum.

Continuity

Continuous Functions on Closed, Finite Intervals



f is defined on $[0, 1]$, but has neither minimum, nor maximum.

The Intermediate Value Theorem

Continuity

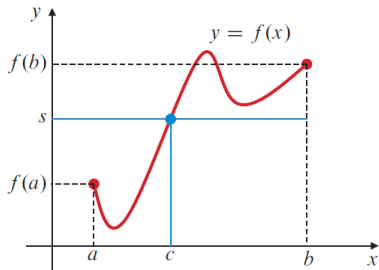
The Intermediate Value Theorem

Theorem.

If f is continuous on a closed, finite interval $[a, b]$, and if s is any number between $f(a)$ and $f(b)$, then there exists at least one point c in (a, b) such that

$$f(c) = s.$$

In other words, the function f takes on every value between $f(a)$ and $f(b)$ at least once in the interval (a, b) .



Continuity

The Intermediate Value Theorem

Determining signs of a continuous function

Between two consecutive roots of a continuous function, the function must take on either all positive or all negative values, but not both.

More precisely, if f is continuous on an interval containing c and d such that $f(c) = f(d) = 0$, but $f(x) \neq 0$ for all x in the open interval (c, d) , then either $f(x) > 0$ or $f(x) < 0$ for all x in (c, d) .

Continuity

The Intermediate Value Theorem

Example

Determine the intervals on which the function $f(x) = x^3 - 4x$ is positive and negative.

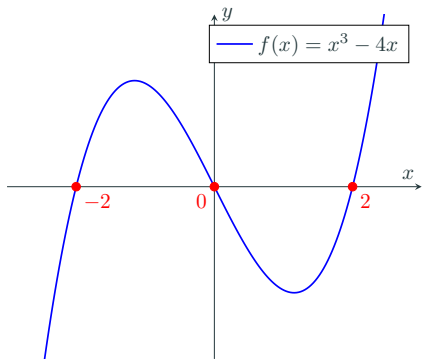
Continuity

The Intermediate Value Theorem

Example

Determine the intervals on which the function $f(x) = x^3 - 4x$ is positive and negative.

Solution



	-2	0	2	
x	$-$	$- \circ$	$+$	$+$
$x^2 - 4$	$+$	$\circ -$	$- \circ$	$+$
$x^3 - 4x$	$-$	$+$	$-$	$+$

Continuity

The Intermediate Value Theorem

Example

Show that the equation $x^3 - x - 1 = 0$ has a root in the interval $[1, 2]$.

Continuity

The Intermediate Value Theorem

Example

Show that the equation $x^3 - x - 1 = 0$ has a root in the interval $[1, 2]$.

Solution

We can apply the Intermediate Value Theorem. First, we evaluate the function at the endpoints of the interval:

$$f(1) = 1^3 - 1 - 1 = -1,$$

$$f(2) = 2^3 - 2 - 1 = 5.$$

Since $f(1) < 0$ and $f(2) > 0$, and f is continuous on $[1, 2]$, there must be at least one root c in $(1, 2)$ such that $f(c) = 0$.

Continuity

The Intermediate Value Theorem

Example (The Bisection Method)

Solve the equation $x^3 - x - 1 = 0$ using the bisection method with an initial interval $[1, 2]$ correct to three decimal places.

Continuity

The Intermediate Value Theorem

Example (The Bisection Method)

Solve the equation $x^3 - x - 1 = 0$ using the bisection method with an initial interval $[1, 2]$ correct to three decimal places.

Solution

Bisection Number	x	$f(x)$	Root in Interval	Midpoint
	1	-1		
	2	5	$[1, 2]$	1.5
1	1.5	0.8750	$[1, 1.5]$	1.25
2	1.25	-0.2969	$[1.25, 1.5]$	1.375
3	1.375	0.2246	$[1.25, 1.375]$	1.3125
4	1.3125	-0.0515	$[1.3125, 1.375]$	1.3438
5	1.3438	0.0826	$[1.3125, 1.3438]$	1.3282
6	1.3282	0.0147	$[1.3125, 1.3282]$	1.3204
7	1.3204	-0.0186	$[1.3204, 1.3282]$	1.3243
8	1.3243	-0.0018	$[1.3243, 1.3282]$	1.3263
9	1.3263	0.0065	$[1.3243, 1.3263]$	1.3253
10	1.3253	0.0025	$[1.3243, 1.3253]$	1.3248
11	1.3248	0.0003	$[1.3243, 1.3248]$	1.3246
12	1.3246	-0.0007	$[1.3246, 1.3248]$	

The root is 1.325, rounded to 3 decimal places.

The Precise Definition of Limit

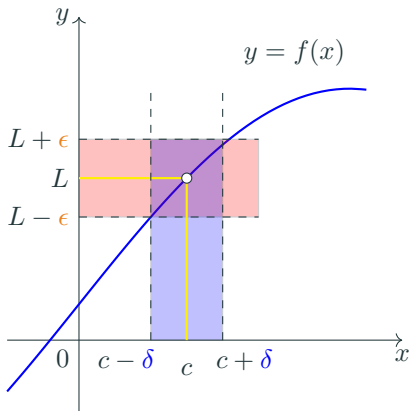
The Precise Definition of Limit

Definition.

Let $f : A \rightarrow \mathbb{R}$ be a function defined on a set A containing the point c . We say that

$$\lim_{x \rightarrow c} f(x) = L$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $|f(x) - L| < \epsilon$.



The Precise Definition of Limit

Example

Example

We will verify that

- $\lim_{x \rightarrow a} x = a$, and
- $\lim_{x \rightarrow a} k = k$.

for any constants a and k .

Solution

We will prove each limit using the epsilon-delta definition.

The Precise Definition of Limit

Example

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We will verify that

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Solution

- *For $\lim_{x \rightarrow a} x = a$:*

The Precise Definition of Limit

Example

Example

We will verify that

- $\lim_{x \rightarrow a} x = a$, and
- $\lim_{x \rightarrow a} k = k$.

for any constants a and k .

Solution

- *For $\lim_{x \rightarrow a} x = a$:*
 - *Let $\epsilon > 0$ be given.*

The Precise Definition of Limit

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We will verify that

- $\lim_{x \rightarrow a} x = a$, and
- $\lim_{x \rightarrow a} k = k$.

for any constants a and k .

Solution

- *For $\lim_{x \rightarrow a} x = a$:*
 - *Let $\epsilon > 0$ be given.*
 - *We are looking for a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|x - a| < \epsilon$.*

The Precise Definition of Limit

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 - *Let $\epsilon > 0$ be given.*
 - *We are looking for a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|x - a| < \epsilon$.*
 - *So, choose $\delta = \epsilon$.*

The Precise Definition of Limit

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- *For $\lim_{x \rightarrow a} x = a$:*
 - *Let $\epsilon > 0$ be given.*
 - *We are looking for a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|x - a| < \epsilon$.*
 - *So, choose $\delta = \epsilon$.*
 - *Then, if $0 < |x - a| < \delta$, we have:*

$$|f(x) - L| = |x - a| < \epsilon.$$

The Precise Definition of Limit

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The Precise Definition of Limit

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The Precise Definition of Limit

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for any constants a and k .

Solution

- *For $\lim_{x \rightarrow a} k = k$:*
 - *Let $\epsilon > 0$ be given.*
 - *We need to find a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.*

The Precise Definition of Limit

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We will verify that

- $\lim_{x \rightarrow a} x = a$, and
- $\lim_{x \rightarrow a} k = k$.

for any constants a and k .

Solution

- *For $\lim_{x \rightarrow a} k = k$:*
 - *Let $\epsilon > 0$ be given.*
 - *We need to find a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.*
 - *Since $f(x) = k$ for all x , we have $|f(x) - L| = |k - k| = 0$.*

The Precise Definition of Limit

Example

Example

We will verify that

- $\lim_{x \rightarrow a} x = a$, and
- $\lim_{x \rightarrow a} k = k$.

for any constants a and k .

Solution

- *For $\lim_{x \rightarrow a} k = k$:*
 - *Let $\epsilon > 0$ be given.*
 - *We need to find a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.*
 - *Since $f(x) = k$ for all x , we have $|f(x) - L| = |k - k| = 0$.*
 - *Thus, we can choose any $\delta > 0$.*

The Precise Definition of Limit

Example

Example

We will verify that

- $\lim_{x \rightarrow a} x = a$, and
- $\lim_{x \rightarrow a} k = k$.

for any constants a and k .

Solution

- *For $\lim_{x \rightarrow a} k = k$:*
 - *Let $\epsilon > 0$ be given.*
 - *We need to find a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.*
 - *Since $f(x) = k$ for all x , we have $|f(x) - L| = |k - k| = 0$.*
 - *Thus, we can choose any $\delta > 0$.*
 - *Then, if $0 < |x - a| < \delta$, we have:*

$$|f(x) - L| = |k - k| = 0 < \epsilon.$$

The Precise Definition of Limit

Example

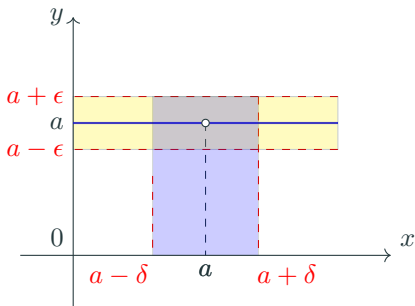
Example

We will verify that

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for any constants a and k .

Solution



The Precise Definition of Limit

Example

Let $f(x) = 3x - 4$. Let us show that $\lim_{x \rightarrow 2} f(x) = 2$.

The Precise Definition of Limit

Example

Let $f(x) = 3x - 4$. Let us show that $\lim_{x \rightarrow 2} f(x) = 2$.

- For every $\epsilon > 0$, we need to find a $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|f(x) - 2| < \epsilon$.

The Precise Definition of Limit

Example

Let $f(x) = 3x - 4$. Let us show that $\lim_{x \rightarrow 2} f(x) = 2$.

- For every $\epsilon > 0$, we need to find a $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|f(x) - 2| < \epsilon$.
- We have $|f(x) - 2| = |3x - 4 - 2| = |3x - 6| = 3|x - 2|$.

The Precise Definition of Limit

Example

Let $f(x) = 3x - 4$. Let us show that $\lim_{x \rightarrow 2} f(x) = 2$.

- For every $\epsilon > 0$, we need to find a $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|f(x) - 2| < \epsilon$.
- We have $|f(x) - 2| = |3x - 4 - 2| = |3x - 6| = 3|x - 2|$.
- Thus, if we choose $\delta = \frac{\epsilon}{3}$, then whenever $0 < |x - 2| < \delta$, we have:

$$|f(x) - 2| = 3|x - 2| < 3\delta = \epsilon.$$

The Precise Definition of Limit

Example

Let $f(x) = 3x - 4$. Let us show that $\lim_{x \rightarrow 2} f(x) = 2$.

The Precise Definition of Limit

Example

Let $f(x) = 9 - x^2$. Let us show that $\lim_{x \rightarrow 2} f(x) = 5$.

The Precise Definition of Limit

Example

Let $f(x) = 9 - x^2$. Let us show that $\lim_{x \rightarrow 2} f(x) = 5$.

- For every $\epsilon > 0$, we need to find a $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|f(x) - 5| < \epsilon$.

The Precise Definition of Limit

Example

Let $f(x) = 9 - x^2$. Let us show that $\lim_{x \rightarrow 2} f(x) = 5$.

- For every $\epsilon > 0$, we need to find a $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|f(x) - 5| < \epsilon$.
- We have $|f(x) - 5| = |9 - x^2 - 5| = |4 - x^2| = |(2 - x)(2 + x)|$.

The Precise Definition of Limit

Example

Let $f(x) = 9 - x^2$. Let us show that $\lim_{x \rightarrow 2} f(x) = 5$.

- For every $\epsilon > 0$, we need to find a $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|f(x) - 5| < \epsilon$.
- We have $|f(x) - 5| = |9 - x^2 - 5| = |4 - x^2| = |(2 - x)(2 + x)|$.
- Thus, if we choose $\delta = \min\{1, \frac{\epsilon}{5}\}$, then whenever $0 < |x - 2| < \delta$, we have:

$$|f(x) - 5| = |(2 - x)(2 + x)| < 5\delta < \epsilon.$$

The Precise Definition of Limit

Example

Let $f(x) = 9 - x^2$. Let us show that $\lim_{x \rightarrow 2} f(x) = 5$.

One-Sided Limits

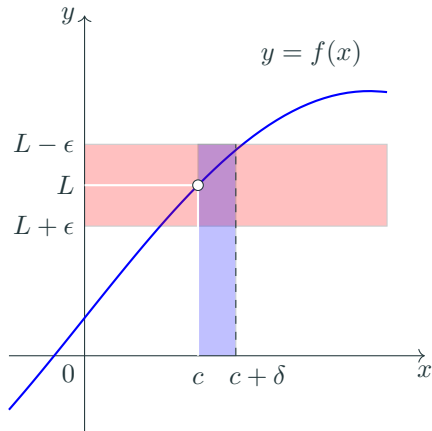
The Precise Definition: One-Sided Limits

Definition.

Let $f : A \rightarrow \mathbb{R}$ be a function defined on a set A containing an interval (c, d) for some $c < d$. We say that

$$\lim_{x \rightarrow c^+} f(x) = L$$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $c < x < c + \delta$, then $|f(x) - L| < \epsilon$.



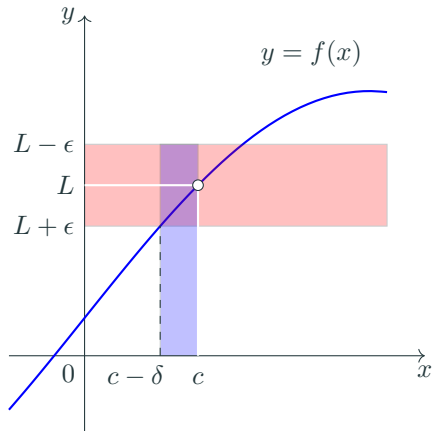
The Precise Definition: One-Sided Limits

Definition.

Let $f : A \rightarrow \mathbb{R}$ be a function defined on a set A containing an interval (b, c) for some $c < d$. We say that

$$\lim_{x \rightarrow c^-} f(x) = L$$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $c - \delta < x < c$, then $|f(x) - L| < \epsilon$.



The Precise Definition: One-Sided Limits

Example

Example

Let $f(x) = \sqrt{x}$. We want to show that $\lim_{x \rightarrow 0^+} f(x) = 0$.

The Precise Definition: One-Sided Limits

Example

Example

Let $f(x) = \sqrt{x}$. We want to show that $\lim_{x \rightarrow 0^+} f(x) = 0$.

Proof.

- Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that if $0 < x < \delta$, then $|f(x) - 0| < \epsilon$.



The Precise Definition: One-Sided Limits

Example

Example

Let $f(x) = \sqrt{x}$. We want to show that $\lim_{x \rightarrow 0^+} f(x) = 0$.

Proof.

- Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that if $0 < x < \delta$, then $|f(x) - 0| < \epsilon$.
- Since $f(x) = \sqrt{x}$, we have $|f(x) - 0| = \sqrt{x}$. Thus, we need to find δ such that if $0 < x < \delta$, then $\sqrt{x} < \epsilon$.



The Precise Definition: One-Sided Limits

Example

Example

Let $f(x) = \sqrt{x}$. We want to show that $\lim_{x \rightarrow 0^+} f(x) = 0$.

Proof.

- Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that if $0 < x < \delta$, then $|f(x) - 0| < \epsilon$.
- Since $f(x) = \sqrt{x}$, we have $|f(x) - 0| = \sqrt{x}$. Thus, we need to find δ such that if $0 < x < \delta$, then $\sqrt{x} < \epsilon$.
- Squaring both sides, we get $x < \epsilon^2$. Therefore, we can choose $\delta = \epsilon^2$.



The Precise Definition: One-Sided Limits

Example

Example

Let $f(x) = \sqrt{x}$. We want to show that $\lim_{x \rightarrow 0^+} f(x) = 0$.

Proof.

- Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that if $0 < x < \delta$, then $|f(x) - 0| < \epsilon$.
- Since $f(x) = \sqrt{x}$, we have $|f(x) - 0| = \sqrt{x}$. Thus, we need to find δ such that if $0 < x < \delta$, then $\sqrt{x} < \epsilon$.
- Squaring both sides, we get $x < \epsilon^2$. Therefore, we can choose $\delta = \epsilon^2$.
- Now, if $0 < x < \delta$, then $0 < x < \epsilon^2$, which implies $\sqrt{x} < \epsilon$. This shows that $\lim_{x \rightarrow 0^+} f(x) = 0$.

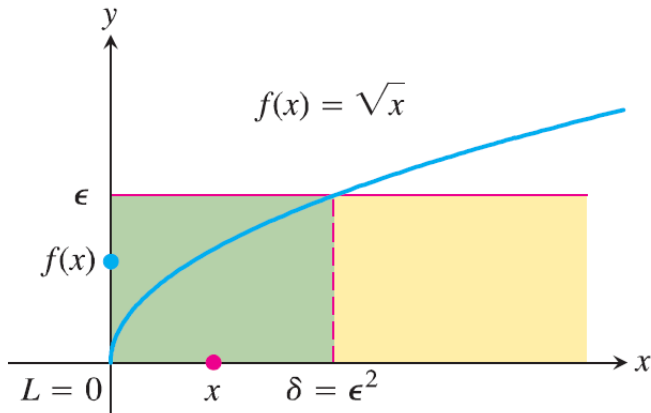


The Precise Definition: One-Sided Limits

Example

Example

Let $f(x) = \sqrt{x}$. We want to show that $\lim_{x \rightarrow 0^+} f(x) = 0$.



Limits at Infinity

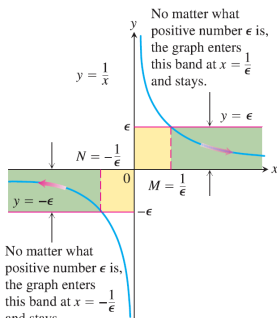
The Precise Definition: Limits at Infinity

Definition.

Let $f : A \rightarrow \mathbb{R}$ be a function defined on a set A containing an interval (a, ∞) for some a . We say that

$$\lim_{x \rightarrow \infty} f(x) = L$$

if for every $\epsilon > 0$, there exists $M > 0$ such that if $x > M$, then $|f(x) - L| < \epsilon$.



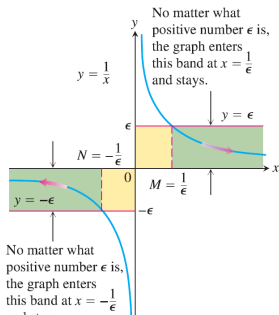
Limits at Infinity

Definition.

Let $f : A \rightarrow \mathbb{R}$ be a function defined on a set A containing an interval $(-\infty, b)$ for some b . We say that

$$\lim_{x \rightarrow -\infty} f(x) = L$$

if for every $\epsilon > 0$, there exists $M < 0$ such that if $x < M$, then $|f(x) - L| < \epsilon$.



Infinite Limits

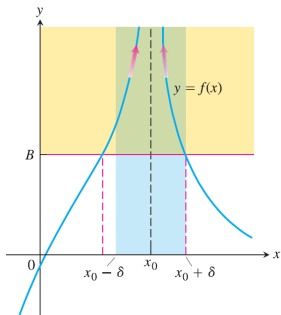
The Precise Definition: Infinite Limits

Definition.

Let $f : A \rightarrow \mathbb{R}$ be a function defined on a set A containing an open neighborhood of x_0 . We say that

$$\lim_{x \rightarrow x_0} f(x) = \infty$$

if for every $B > 0$, there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $f(x) > B$.



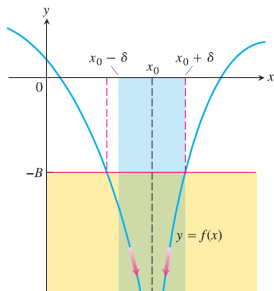
The Precise Definition: Infinite Limits

Definition.

Let $f : A \rightarrow \mathbb{R}$ be a function defined on a set A containing an open neighborhood of x_0 . We say that

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The Precise Definition: Infinite Limits

Example

Verify that $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$.

The Precise Definition: Infinite Limits

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- *Let $B > 0$ be given. We need to find a $\delta > 0$ such that if $0 < |x| < \delta$, then $\frac{1}{x^2} > B$.*
- *This is equivalent to $x^2 < \frac{1}{B}$, or $|x| < \frac{1}{\sqrt{B}}$.*

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- Let $B > 0$ be given. We need to find a $\delta > 0$ such that if $0 < |x| < \delta$, then $\frac{1}{x^2} > B$.
- This is equivalent to $x^2 < \frac{1}{B}$, or $|x| < \frac{1}{\sqrt{B}}$.
- Thus, we can choose $\delta = \frac{1}{\sqrt{B}}$.
- Now, if $0 < |x| < \delta$, then $0 < |x| < \frac{1}{\sqrt{B}}$, which implies $\frac{1}{x^2} > B$.

Conclusion

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- These concepts are crucial for further studies in calculus, particularly in understanding continuity and differentiability of functions.

Conclusion

In this lecture, we have explored

- the precise definitions of limits,
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- We have also verified these definitions through examples and solutions,
- providing a solid foundation for understanding limits in calculus.
- These concepts are crucial for further studies in calculus, particularly in understanding continuity and differentiability of functions.
- Now, we are ready to explore the concept of differentiability of functions.

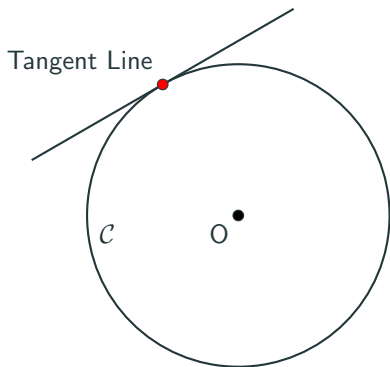
Differentiation

Differentiation

Tangent Lines to Curves

A tangent line to a circle has the following properties:

- (i) It touches the circle at exactly one point.

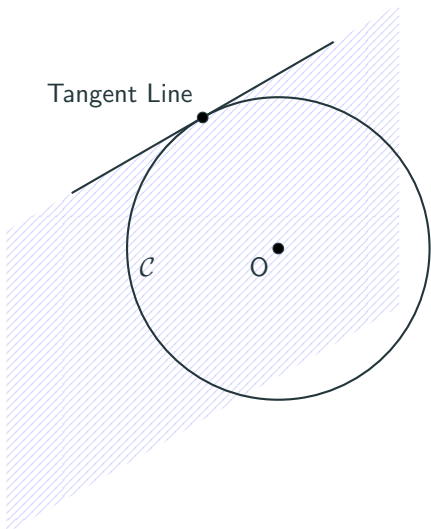


Differentiation

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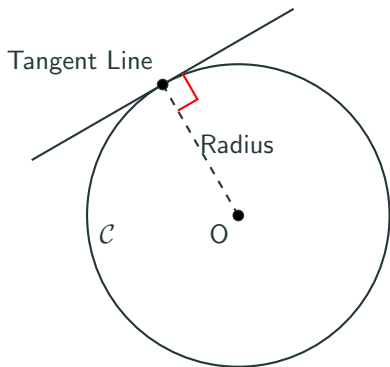


Differentiation

Tangent Lines to Curves

A tangent line to a circle has the following properties:

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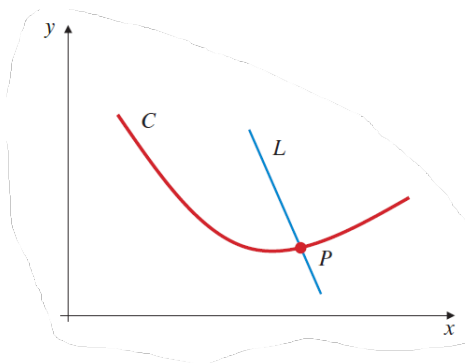


Differentiation

Tangent Lines to Curves

But these properties are not sufficient to define a tangent line to a general curve.

- (i) It touches the **curve** at exactly one point.



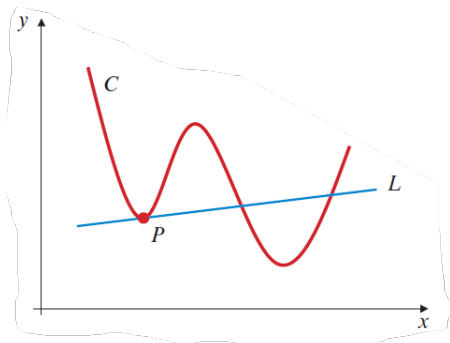
L meets C only at P but is not tangent to C .

Differentiation

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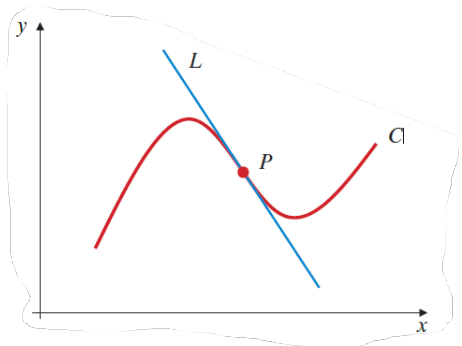
L meets C at several points but is tangent to C at P .

Differentiation

Tangent Lines to Curves

But these properties are not sufficient to define a tangent line to a general curve.

- (i) It touches the **curve** at exactly one point.
- (ii) The **curve** lies entirely on one side of the tangent line.



L is tangent to C at P but the curve lies both sides of L .

Differentiation

Tangent Lines to Curves

But these properties are not sufficient to define a tangent line to a general curve.

- (i) It touches the **curve** at exactly one point.
- (ii) The **curve** lies entirely on one side of the tangent line.
- (iii) It is perpendicular to the **radius** at the point of contact.



Many curves do not have a radius, so the concept of perpendicularity does not apply.

Differentiation

Tangent Lines to Curves

The slope of the tangent line at point P is the limit of the slope of the secant lines as point Q approaches point P .

Definition. *Nonvertical tangent lines*

Suppose that the function f is continuous at $x = x_0$ and that

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. Then the line through the point $P = (x_0, f(x_0))$ with slope equal to this limit is called the **tangent line** to the graph of f at the point P .

Differentiation

Example

Find an equation of the tangent line to the curve $y = x^2$ at the point $(1, 1)$.

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Solution

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Differentiation

Example

Find an equation of the tangent line to the curve $y = x^2$ at the point $(1, 1)$.

Solution

- *Let $f(x) = x^2$.*
- *We find the slope of the tangent line at the point $(1, 1)$.*

Differentiation

Example

Find an equation of the tangent line to the curve $y = x^2$ at the point $(1, 1)$.

Solution

- Let $f(x) = x^2$.
- We find the slope of the tangent line at the point $(1, 1)$.
- We have

$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \rightarrow 0} \frac{2h + h^2}{h} = 2.$$

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- The equation of the tangent line in point-slope form is

$$y - 1 = 2(x - 1).$$

Simplifying this, we get

$$y = 2x - 1.$$

Definition. *Vertical tangents*

If f is continuous at $P = (x_0, f(x_0))$, and if either

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \infty$$

or

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = -\infty,$$

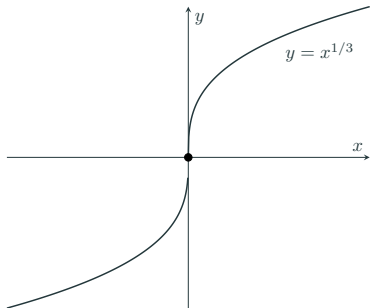
then the vertical line $x = x_0$ is called the **tangent** to the graph of f at the point P .

If the above limit fails to exist in any other way than being ∞ or $-\infty$, then the graph $y = f(x)$ has no tangent line at P .

Differentiation

Example

Let $f(x) = \sqrt[3]{x} = x^{1/3}$. We look for the tangent line (if exists) to the curve $y = f(x)$ at the origin $(0, 0)$.

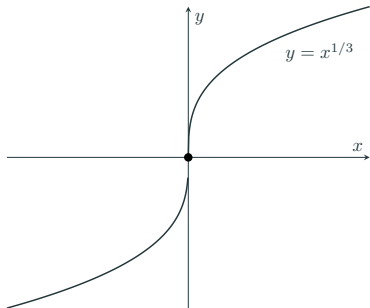


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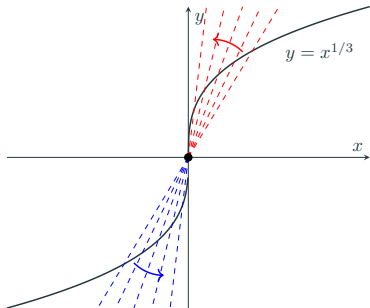
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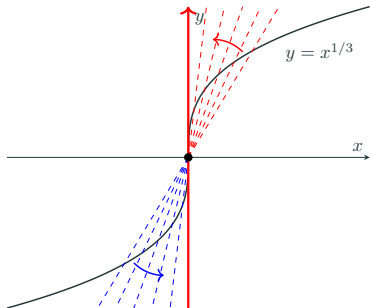
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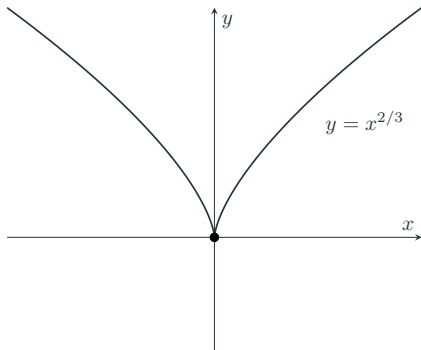
- As h approaches 0 from either side, this limit goes to ∞ .



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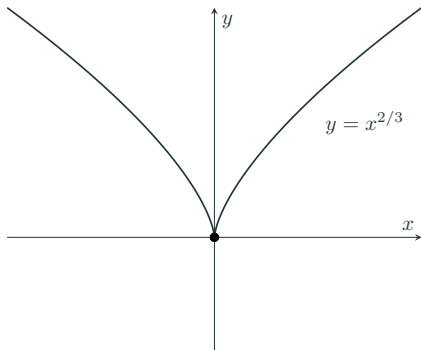


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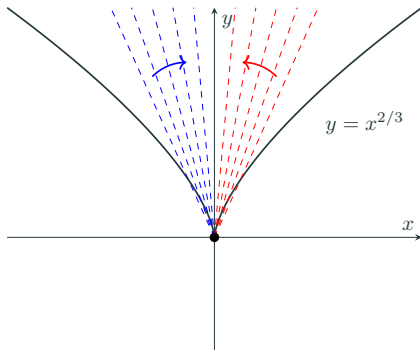
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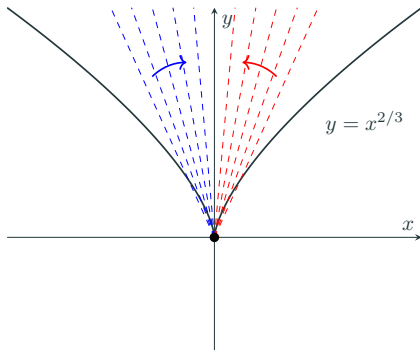
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- As h approaches 0 from left, this limit goes to $-\infty$ while as h approaches 0 from right, this limit goes to $+\infty$.

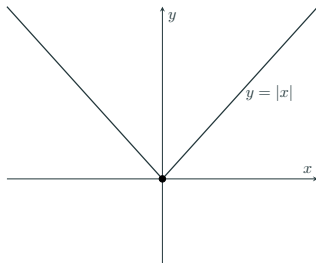


Differentiation

Example

Does the graph of $y = |x|$ have a tangent line at $x = 0$?

Solution



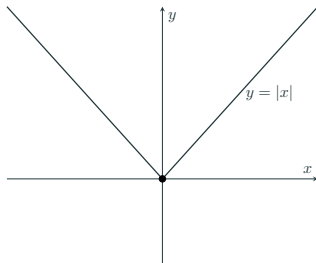
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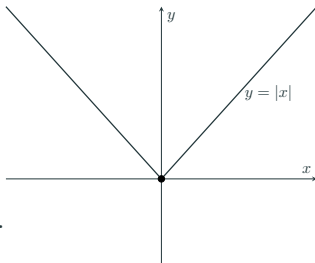
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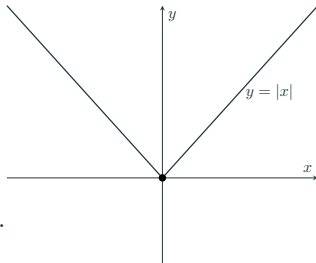
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- Since the left-hand limit and right-hand limit are not equal, the limit does not exist.

Definition. The slope of a curve

Let f be a function defined on an open interval containing x_0 . The slope of the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is defined as

$$m = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided that this limit exists. In this case, the curve has a **tangent line** at the point $(x_0, f(x_0))$ with slope m .

Definition. *Normal line*

The **normal line** to the curve at the point $(x_0, f(x_0))$ is the line that is perpendicular to the tangent line at that point. If the slope of the tangent line is m , then the slope of the normal line is given by

$$m_{\text{normal}} = -\frac{1}{m},$$

provided that $m \neq 0$.