MAT123 MATHEMATICS I

Lecture 08: Differentiation

Outline

The Derivative

Differentiation

Definition. *The derivative*

The **derivative** of a function f at a point x_0 is defined as

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided that this limit exists. In this case, we say that f is **differentiable** at x_0.

If x is in the domain of f and f is not differentiable at x, we say that x is a **singular point** of f.

Differentiation

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Derivative as a Function

One can think of the derivative as a defining rule for another function, called the **derivative function** or simply the **derivative** of f, denoted by f'. Note that the domain of f' is the set of all points in the domain of f at which f is differentiable.

Remark

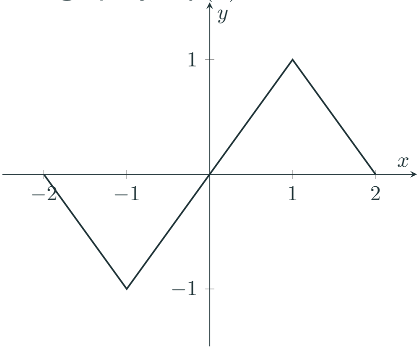
The value of the derivative of f at a particular point x_0 can be expressed as a limit in either of two ways:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

In the second limit, $x_0 + h$ is replaced by x, which approaches x_0 as h approaches 0.

Differentiation

The graph $y = f(x)$



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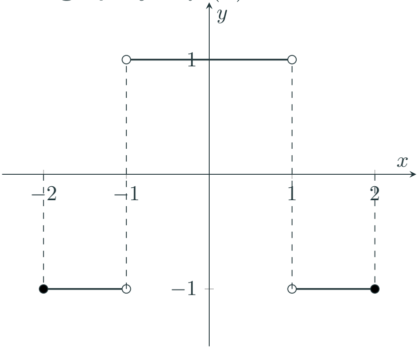

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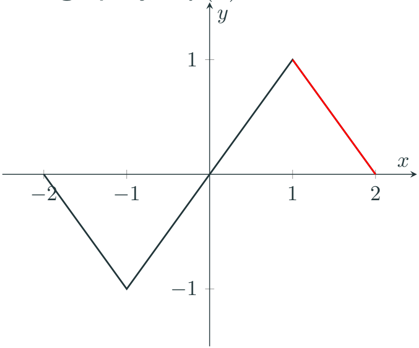

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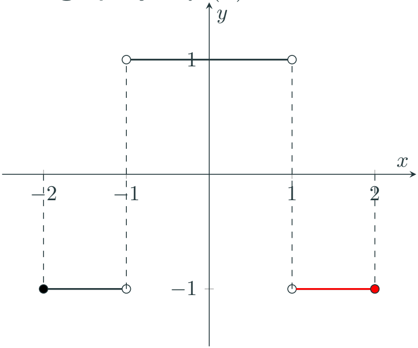

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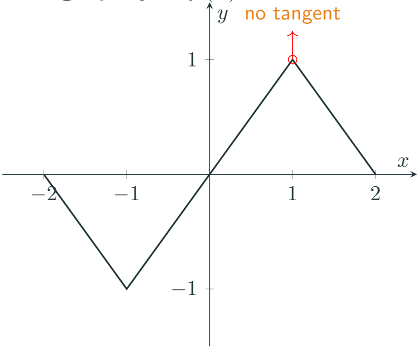

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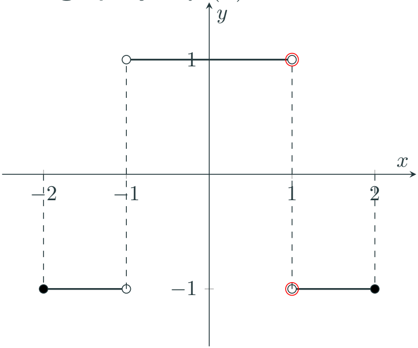

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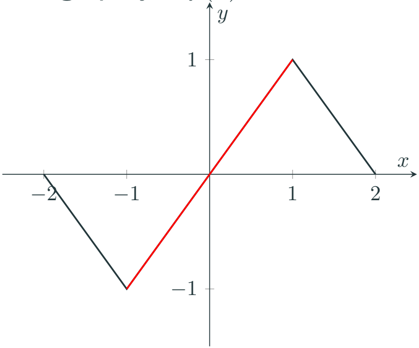

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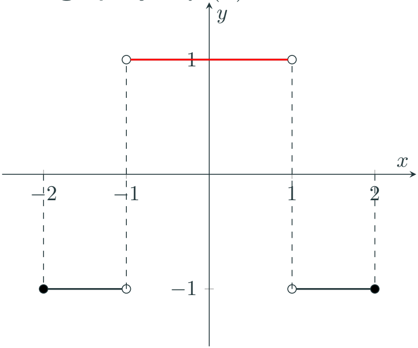

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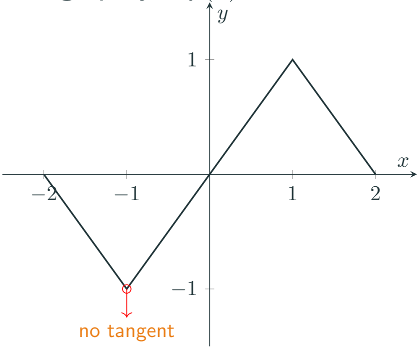

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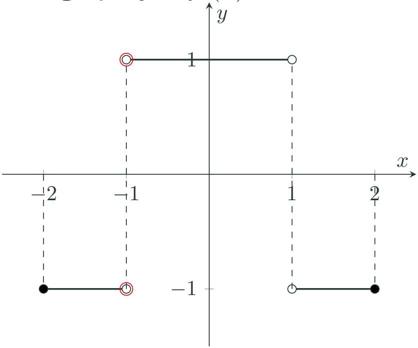

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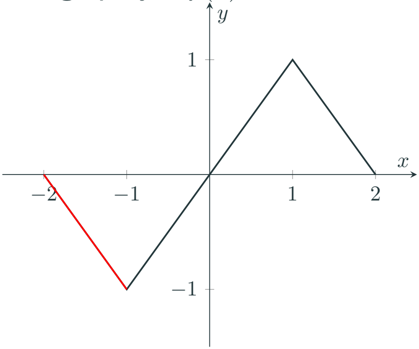

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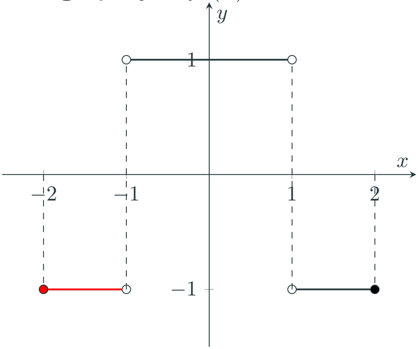

Definition. *One-sided derivative*

Let $f : A \to \mathbb{R}$ be a function defined on a set A containing the point c. The right-hand derivative of f at c, denoted $f'_+(c)$, is defined as:

$$f'_+(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

if this limit exists.

Similarly, the left-hand derivative of f at c, denoted $f'_-(c)$, is defined as:

$$f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

if this limit exists.

The Derivative

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We now say that f is **differentiable** on $[a, b]$ if f is differentiable on (a, b) and both $f'_+(a)$ and $f'_-(b)$ exist.

Example (The derivative of a linear function)

Show that if $f(x) = ax + b$, then $f'(x) = a$.

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Solution

We have:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a(x+h) + b - (ax+b)}{h} = \lim_{h \to 0} \frac{ah}{h} = a.$$

Example

Use the definition of the derivative to calculate the derivatives of the following functions:

(a) $f(x) = x^2$ **(b)** $g(x) = \dfrac{1}{x}$ **(c)** $h(x) = \sqrt{x}$

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Solution

(a) *We have:*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{2xh + h^2}{h}$$
$$= \lim_{h\to 0} (2x + h) = 2x.$$

Example

Use the definition of the derivative to calculate the derivatives of the following functions:

(a) $f(x) = x^2$ **(b)** $g(x) = \dfrac{1}{x}$ **(c)** $h(x) = \sqrt{x}$

Solution

(b) We have:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$
$$= \lim_{h\to 0} \frac{x - (x+h)}{h(x+h)x}$$
$$= \lim_{h\to 0} \frac{-h}{h(x+h)x}$$
$$= \lim_{h\to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}.$$

The Derivative

Example

Use the definition of the derivative to calculate the derivatives of the following functions:

(a) $f(x) = x^2$ **(b)** $g(x) = \dfrac{1}{x}$ **(c)** $h(x) = \sqrt{x}$

Solution

(c) We have:

$$\begin{aligned} h'(x) = \lim_{h\to 0} \frac{h(x+h) - h(x)}{h} &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

Theorem *(General Power Rule)*

If $f(x) = x^n$ for some real number n, then $f'(x) = nx^{n-1}$.
This formula is valid for <u>all values</u> of n for which x^{n-1} makes sense as a real number.

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Example (Differentiating powers)

If $f(x) = x^{5/3}$, then by the General Power Rule:

$$f'(x) = \frac{5}{3}x^{5/3-1} = \frac{5}{3}x^{2/3}.$$

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If $g(t) = \frac{1}{\sqrt{t}}$, then $g'(t) = -\frac{1}{2}t^{-3/2}$ for all real $t > 0$.

Example (Differentiating the absolute value function)

Show that if $f(x) = |x|$, then $f'(x)$ exists for all $x \neq 0$ and is given by:

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

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Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{|x+h| - |x|}{h}.$$

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Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{|x+h| - |x|}{h}.$$

For $x > 0$ and sufficiently small $|h|$, we have $|x+h| = x + h$ and $|x| = x$, so:

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For $x < 0$ and sufficiently small $|h|$, we have $|x+h| = -(x+h)$ and $|x| = -x$, so:

$$f'(x) = \lim_{h \to 0} \frac{-(x+h) + x}{h} = \lim_{h \to 0} \frac{-h}{h} = -1.$$

The Derivative

Leibniz Notation

Given a function $y = f(x)$, the derivative of f with respect to x is denoted by one of the following notation:

- $f'(x)$
- $\frac{dy}{dx}$
- $D_x f(x)$
- y'
- $\frac{d}{dx} f(x)$

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- $\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dt} t^{100} = 100t^{99}$

The Derivative

The value of the derivative of a function f at a point x_0 can also be expressed in several ways:

- $D_x y\Big|_{x=x_0}$
- $f'(x_0)$
- $\dfrac{dy}{dx}\Big|_{x=x_0}$
- $\dfrac{d}{dx}f(x)\Big|_{x=x_0}$
- $y'\Big|_{x=x_0}$
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- $y'\Big|_{x=x_0}$
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$$\begin{aligned}\frac{d}{dx}x^4\Big|_{x=-1} &= 4x^3\Big|_{x=-1} \\ &= 4(-1)^3 = -4\end{aligned}$$

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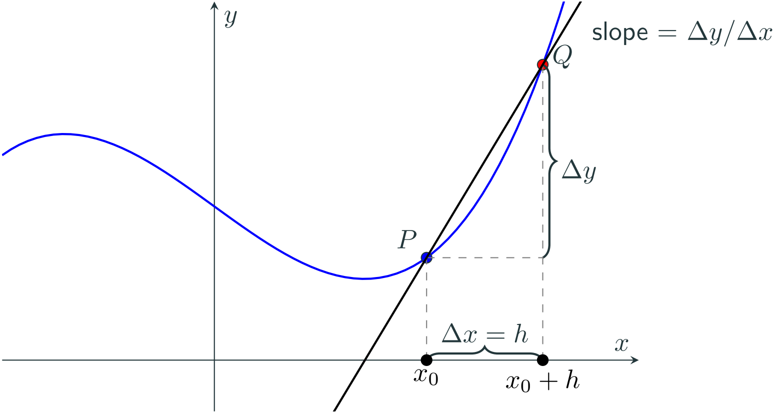
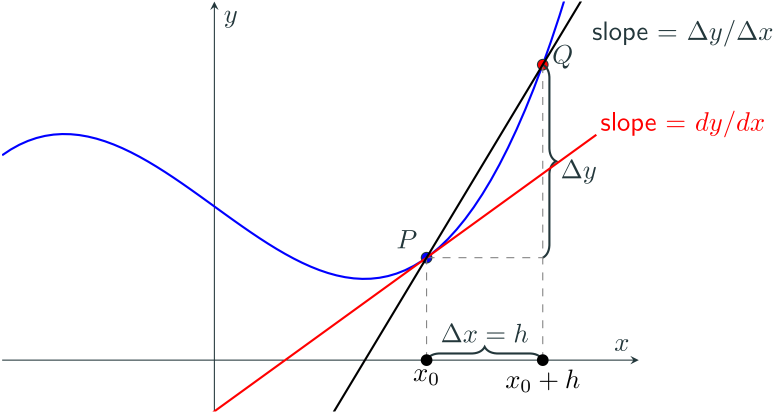
The Derivative

Leibniz Notation





$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

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$$dy = \frac{dy}{dx}\,dx = f'(x)\,dx$$

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For example, if $f(x) = x^2$, then:

$$dy = \frac{dy}{dx}\,dx = f'(x)\,dx = 2x\,dx.$$

Differentiation Rules

Theorem *(Differentiability implies continuity)*

If f is differentiable at a point c, then f is continuous at c.

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If f is differentiable at a point c, then f is continuous at c.

Remark

The converse may not be true: a function can be continuous at a point without being differentiable there. For example, the function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable there.

Theorem *(Differentiation rules for sums, differences, and constant multiples)*

If functions f and g are differentiable at a point x, then the functions $f + g$, $f - g$, and $c \cdot f$ (where c is a constant) are also differentiable at that point, and their derivatives are given by:

- $(f + g)' = f' + g'$,
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Proof.

Let f and g be differentiable at x. Then, by the definition of the derivative:

$$(f+g)'(x) = \lim_{h \to 0} \frac{(f(x+h)+g(x+h)) - (f(x)+g(x))}{h}$$

$$= \lim_{h \to 0} \left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \right) = f'(x) + g'(x).$$

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Proof.

The proofs for the difference and constant multiple rules are similar. □

Differentiation Rules

Example

Calculate the derivatives of the functions:

(a) $f(x) = 2x^3 - 5x^2 + 4x + 7$

(b) $f(x) = 5\sqrt{x} + \frac{3}{x} - 18$

(c) $y = \frac{1}{7}t^4 - 3t^{7/3}$

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Solution

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Solution

(a) $f'(x) = 6x^2 - 10x + 4$

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Solution

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(b) $f'(x) = \frac{5}{2\sqrt{x}} - \frac{3}{x^2}$

Example

Calculate the derivatives of the functions:

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(c) $y = \frac{1}{7}t^4 - 3t^{7/3}$

Solution

(a) $f'(x) = 6x^2 - 10x + 4$

(b) $f'(x) = \frac{5}{2\sqrt{x}} - \frac{3}{x^2}$

(c) $y' = \frac{4}{7}t^3 - 3 \cdot \frac{7}{3}t^{4/3} = \frac{4}{7}t^3 - 7t^{4/3}$

Example

Find an equation of the tangent line to the curve

$$y = \frac{3x^3 - 4}{x}$$

at the point on the curve where $x = -2$.

Solution

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To find the equation of the tangent line, we need to find the derivative of the function and evaluate it at the point $x = -2$.

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Solution

To find the equation of the tangent line, we need to find the derivative of the function and evaluate it at the point $x = -2$.

We may rewrite y:

$$y = \frac{3x^3 - 4}{x} = 3x^2 - \frac{4}{x}.$$

Now, we differentiate:

$$y' = 6x + \frac{4}{x^2}.$$

Example

Find an equation of the tangent line to the curve

$$y = \frac{3x^3 - 4}{x}$$

at the point on the curve where $x = -2$.

Solution

Next, we evaluate the derivative at $x = -2$:

$$y'(-2) = 6(-2) + \frac{4}{(-2)^2} = -12 + 1 = -11.$$

Now we need the point on the curve where $x = -2$:

$$y(-2) = \frac{3(-2)^3 - 4}{-2} = \frac{-24 - 4}{-2} = \frac{-28}{-2} = 14.$$

Example

Find an equation of the tangent line to the curve

$$y = \frac{3x^3 - 4}{x}$$

at the point on the curve where $x = -2$.

Solution

So the point is $(-2, 14)$.

Now we can use the point-slope form of the equation of a line:

$$y - y_1 = m(x - x_1),$$

where $(x_1, y_1) = (-2, 14)$ *and* $m = -11$*:*

$$y - 14 = -11(x + 2).$$

Example

Find an equation of the tangent line to the curve

$$y = \frac{3x^3 - 4}{x}$$

at the point on the curve where $x = -2$.

Solution

Simplifying, we get:

$$y = -11x - 22 + 14 = -11x - 8.$$

Thus, the equation of the tangent line is:

$$y = -11x - 8.$$

Differentiation Rules

Theorem *(The Product Rule)*

If functions f and g are differentiable at x, then their product fg is also differentiable at x, and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

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Proof.
Let f and g be differentiable at x. Then, by the definition of the derivative:

$$\begin{aligned}(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right) \\ &= f'(x)g(x) + f(x)g'(x).\end{aligned}$$

□

Example

Find the derivative of the function $f(x) = (2x^3 - 5)(x^2 + 3)$ using and without using the product rule.

Solution

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Find the derivative of the function $f(x) = (2x^3 - 5)(x^2 + 3)$ using and without using the product rule.

Solution

Without the Product Rule:

$$f(x) = (2x^3 - 5)(x^2 + 3) = 2x^5 + 6x^3 - 5x^2 - 15.$$

Now, we differentiate:

$$f'(x) = 10x^4 + 18x^2 - 10x.$$

Example

Find the derivative of the function $f(x) = (2x^3 - 5)(x^2 + 3)$ using and without using the product rule.

Solution

Using the Product Rule: Let $u = 2x^3 - 5$ *and* $v = x^2 + 3$. *Then,*

$$\begin{aligned} f'(x) &= u'v + uv' \\ &= (6x^2)(x^2 + 3) + (2x^3 - 5)(2x) \\ &= 6x^4 + 18x^2 + 4x^4 - 10x \\ &= 10x^4 + 18x^2 - 10x. \end{aligned}$$

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The product rule can be extended to products of any number of factors, for instance:

$$(f_1 f_2 \cdots f_n)' = f_1' f_2 f_3 \cdots f_n + f_1 f_2' f_3 \cdots f_n + \cdots + f_1 f_2 f_3 \cdots f_n'.$$

Differentiation Rules

Theorem *(The Reciprocal Rule)*

If f is differentiable at x and $f(x) \neq 0$, then the function $g(x) = \frac{1}{f(x)}$ is also differentiable at x, and its derivative is given by:

$$g'(x) = -\frac{f'(x)}{(f(x))^2}.$$

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Proof.
Let f be differentiable at x and $f(x) \neq 0$. Then, by definition of the derivative:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} \\ &= \lim_{h\to 0} \frac{f(x) - f(x+h)}{hf(x)f(x+h)} = \lim_{h\to 0} \left(\frac{-1}{f(x+h)f(x)}\right) \frac{f(x+h) - f(x)}{h} \\ &= \frac{-1}{(f(x))^2} f'(x). \end{aligned}$$

□

Differentiation Rules

Example

Differentiate the functions:

(a) $f(x) = \dfrac{1}{x^2+1}$ (b) $g(t) = \dfrac{1}{t+\frac{1}{t}}$

Solution

Differentiation Rules

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Solution

(a) Using the Reciprocal Rule:

$$f'(x) = -\frac{(x^2+1)'}{(x^2+1)^2} = -\frac{2x}{(x^2+1)^2}.$$

Example

Differentiate the functions:

(a) $f(x) = \dfrac{1}{x^2+1}$ (b) $g(t) = \dfrac{1}{t+\frac{1}{t}}$

Solution

(b) Using the Reciprocal Rule:

$$g'(t) = -\frac{(t+\frac{1}{t})'}{(t+\frac{1}{t})^2} = -\frac{1-\frac{1}{t^2}}{(t+\frac{1}{t})^2} = -\frac{\frac{t^2-1}{t^2}}{(t+\frac{1}{t})^2} = -\frac{t^2-1}{t^2(t+\frac{1}{t})^2}.$$

Differentiation Rules

Theorem *(The Quotient Rule)*

If functions f and g are differentiable at x and $g(x) \neq 0$, then the quotient $\frac{f}{g}$ is also differentiable at x, and its derivative is given by:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

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Proof.
Combining the Product Rule and the Reciprocal Rule, we have:

$$\begin{aligned}\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\cdot\frac{1}{g(x)}\right) &= f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\frac{d}{dx}\left(\frac{1}{g(x)}\right)\\ &= f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\left(-\frac{g'(x)}{(g(x))^2}\right)\\ &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\end{aligned}$$

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Example

Find the derivatives of

(a) $f(x) = \dfrac{x^2+1}{x^3-2}$ (b) $g(t) = \dfrac{\sqrt{t}}{3-5t}$ (c) $h(\theta) = \dfrac{a+b\theta}{m+n\theta}$

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Solution

(a) Using the Quotient Rule:

$$f'(x) = \frac{(x^2+1)'(x^3-2)-(x^2+1)(x^3-2)'}{(x^3-2)^2} = \frac{(2x)(x^3-2)-(x^2+1)(3x^2)}{(x^3-2)^2}.$$

Simplifying, we get:

$$f'(x) = \frac{2x^4-4x-3x^4-3x^2}{(x^3-2)^2} = \frac{-x^4-3x^2-4x}{(x^3-2)^2}.$$

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Solution

(b) Using the Quotient Rule:

$$g'(t) = \frac{(\sqrt{t})'(3-5t) - (\sqrt{t})(3-5t)'}{(3-5t)^2} = \frac{\frac{1}{2\sqrt{t}}(3-5t) - \sqrt{t}(-5)}{(3-5t)^2}.$$

Simplifying, we get:

$$g'(t) = \frac{\frac{3}{2\sqrt{t}} + 5\sqrt{t}}{(3-5t)^2} = \frac{\frac{3+10t}{2\sqrt{t}}}{(3-5t)^2} = \frac{3+10t}{2\sqrt{t}(3-5t)^2}.$$

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Solution

(c) Using the Quotient Rule:

$$h'(\theta) = \frac{(a+b\theta)'(m+n\theta) - (a+b\theta)(m+n\theta)'}{(m+n\theta)^2} = \frac{(b)(m+n\theta) - (a+b\theta)(n)}{(m+n\theta)^2}.$$

Simplifying, we get:

$$h'(\theta) = \frac{bm + bn\theta - an - bn\theta}{(m+n\theta)^2} = \frac{bm - an}{(m+n\theta)^2}.$$

Example

Find equations of any lines that pass through the point $(-1, 0)$ and are tangent to the curve $y = (x-1)/(x+1)$.

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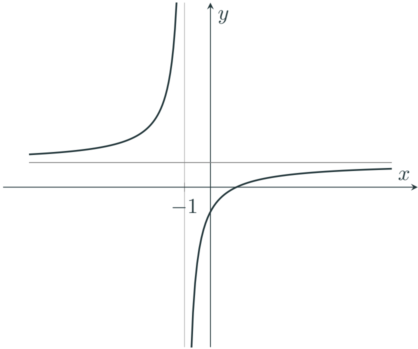
Find equations of any lines that pass through the point $(-1, 0)$ and are tangent to the curve $y = (x - 1)/(x + 1)$.

Solution

- Assume that the tangent meets the curve at the point (a, b).



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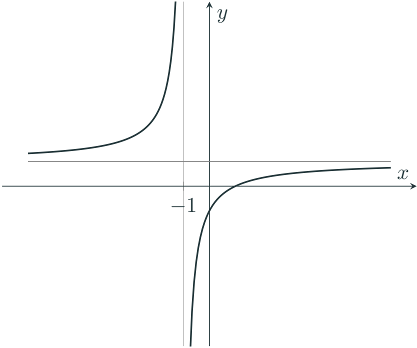
Solution

- Assume that the tangent meets the curve at the point (a, b).
- Since

$$y' = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2},$$

the slope of the tangent is $\frac{2}{(a+1)^2}$.



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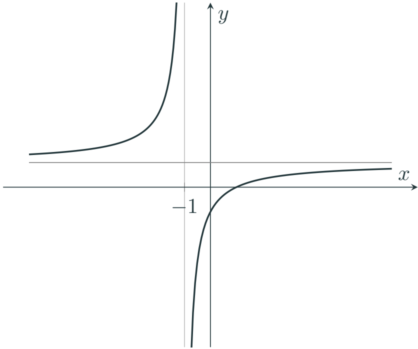
$$\frac{b-0}{a-(-1)} = \frac{b}{a+1}.$$



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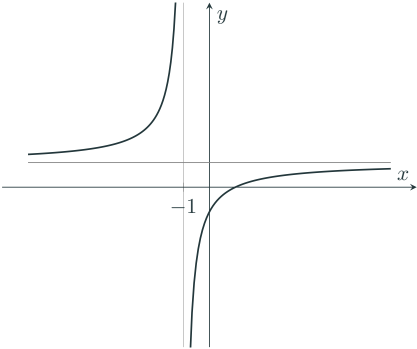

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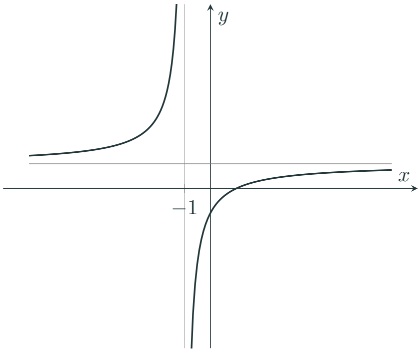

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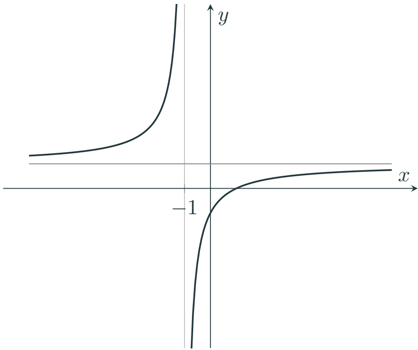

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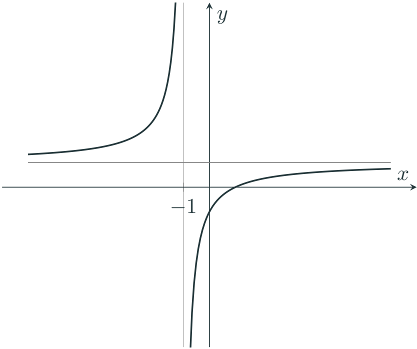

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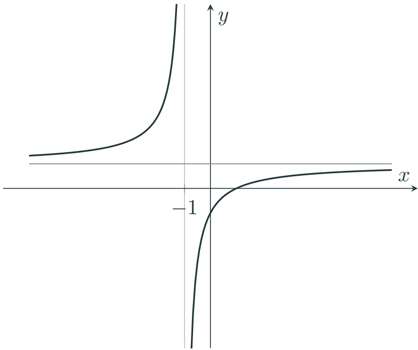

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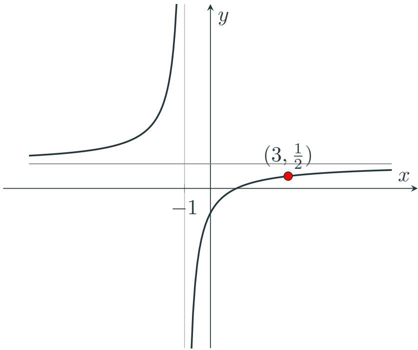
- Thus, the point of tangency is $(3, \frac{1}{2})$.



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Solution

- The equation of the tangent line at the point of tangency is given by

$$y - \frac{1}{2} = \frac{2}{(3+1)^2}(x-3).$$

Simplifying, we find

$$y - \frac{1}{2} = \frac{1}{8}(x-3) \implies y = \frac{1}{8}x + \frac{1}{8}.$$