MAT123 MATHEMATICS I

Lecture 09: Differentiation

Outline

Differentiation Rules

The Chain Rule

Theorem *(The Chain Rule)*

If $f(u)$ is differentiable at $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $y = f(g(x))$ is also differentiable at x, and its derivative is given by:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x),$$

or in the Leibniz notation:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

The Chain Rule

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Find the derivative of $y = \sqrt{x^2 + 1}$.

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First, we find $\frac{dy}{du}$ *and* $\frac{du}{dx}$*:*

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}} \quad \textit{and} \quad \frac{du}{dx} = 2x.$$

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$$\frac{dy}{du} = \frac{1}{2\sqrt{u}} \quad \textit{and} \quad \frac{du}{dx} = 2x.$$

Now, applying the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}.$$

The Chain Rule

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Find the derivatives of the following functions:

(a) $(7x-3)^{10}$ **(b)** $f(t) = |t^2 - 1|$ **(c)** $\left(3x + \frac{1}{(2x+1)^3}\right)^{1/4}$.

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Solution

(a) *Let* $u=7x-3$, *then*
$f(x)=u^{10}$.

$$\begin{aligned} f'(x)=10u^9\cdot u' &= 10(7x-3)^9\cdot 7 \\ &= 70(7x-3)^9. \end{aligned}$$

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$$=70(7x-3)^9.$$

(b) *Let* $u=t^2-1$*, then* $f(t)=|u|$.

$$f'(t)=\frac{d}{dt}|u|=\frac{du}{dt}\cdot\operatorname{sgn}(u)$$
$$=2t\cdot\operatorname{sgn}(t^2-1).$$

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$$f'(t) = \frac{d}{dt}|u| = \frac{du}{dt} \cdot \operatorname{sgn}(u) = 2t \cdot \operatorname{sgn}(t^2 - 1).$$

(c) *Let* $u = 3x + (2x+1)^{-3}$:

$$f'(x) = \frac{1}{4}u^{-3/4}u' = \frac{1}{4}\left(3x + (2x+1)^{-3}\right)^{-3/4}\left(3 - 6(2x+1)^{-4}\right).$$

The Chain Rule

Building the Chain Rule into Differential Formulas

If u is a differentiable function of x and $y = u^n$, then

- $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = nu^{n-1} \cdot \frac{du}{dx} \qquad \text{(the Power Rule)}$$

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- $$\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx} \qquad \text{(the Reciprocal Rule)}$$

Derivatives of Trigonometric Functions

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Theorem *(The derivative of the sine function is the cosine function)*

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$$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}.$$

Simplifying, we get:

$$\begin{aligned}\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} = 0 + \cos x \cdot 1.\end{aligned}$$

Derivatives of Trigonometric Functions

Theorem *(The derivative of the cosine function is the negative sine function)*

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Proof.
We combine the Chain Rule with the fact that $\frac{d}{dx}(\sin x) = \cos x$:

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}(\sin(\frac{\pi}{2} - x)) = -\cos(\frac{\pi}{2} - x) = -\sin x.$$

□

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We can use the product rule and the chain rule:

$$\frac{d}{dx}(x^2 \sin \sqrt{x}) = 2x \sin \sqrt{x} + x^2 \cos \sqrt{x} \cdot \frac{1}{2\sqrt{x}}.$$

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Simplifying gives:

$$= 2x \sin \sqrt{x} + \frac{x^{3/2} \cos \sqrt{x}}{2}.$$

Thus, the derivative is:

$$\frac{d}{dx}(x^2 \sin \sqrt{x}) = 2x \sin \sqrt{x} + \frac{x^{3/2} \cos \sqrt{x}}{2}.$$

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Use two methods to find the derivative of the function $f(t) = \sin t \cdot \cos t$.

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Solution

We can use the product rule:

$$f'(t) = \frac{d}{dt}(\sin t) \cdot \cos t + \sin t \cdot \frac{d}{dt}(\cos t) = \cos t \cdot \cos t - \sin t \cdot \sin t = \cos^2 t - \sin^2 t.$$

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Alternatively, we can use the identity $\sin t \cdot \cos t = \frac{1}{2}\sin(2t)$*:*

$$f'(t) = \frac{1}{2}\frac{d}{dt}(\sin(2t)) = \frac{1}{2} \cdot 2\cos(2t) = \cos(2t).$$

Derivatives of Trigonometric Functions

Theorem *(The Derivatives of the Other Trigonometric Functions)*

- $\dfrac{d}{dx}(\tan x) = \sec^2 x$
- $\dfrac{d}{dx}(\cot x) = -\csc^2 x$
- $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
- $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

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• *First, we need to find the derivative of the function:*

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- *Evaluating this at* $x = 1$*:* $y'(1) = \sec^2\left(\frac{\pi}{4}\right) \cdot \frac{\pi}{4} = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$.

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- *Evaluating this at $x = 1$:* $y'(1) = \sec^2\left(\frac{\pi}{4}\right) \cdot \frac{\pi}{4} = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$.
- *The point on the curve at $x = 1$ is:* $y(1) = \tan\left(\frac{\pi}{4}\right) = 1$.
- *Thus, the equation of the tangent line is:*

$$y - 1 = \tfrac{\pi}{2}(x - 1) \implies y = \tfrac{\pi}{2}x - \tfrac{\pi}{2} + 1.$$

The normal line has a slope that is the negative reciprocal:

$$y - 1 = -\tfrac{2}{\pi}(x - 1) \implies y = -\tfrac{2}{\pi}x + \tfrac{2}{\pi} + 1.$$

Higher-Order Derivatives

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Suppose the function $y = f(x)$ is differentiable at x. Also, suppose that $y' = f'(x)$ is also differentiable at x. Then we can define the second derivative f'' as:

$$y'' = f''(x) = (f'(x))' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right) = \frac{d^2}{dx^2}f(x).$$

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Definition

The n-th derivative of a function f is denoted by $f^{(n)}$ and is defined recursively as follows:

- f' for $n = 1$,
- $f'' = (f')'$ for $n = 2$,
- $f''' = (f'')'$ for $n = 3$,

  $\vdots$
- $f^{(n)} = \left[f^{(n-1)}\right]'$ for $n \geq 2$

Higher-Order Derivatives

Example

The **velocity** of a moving object is the (instantaneous) rate of change of its position with respect to time, which is given by the first derivative of the position function $s(t)$:

$$v(t) = s'(t).$$

The **acceleration** is the rate of change of velocity with respect to time, given by the second derivative:

$$a(t) = v'(t) = s''(t).$$

Higher-Order Derivatives

Example

If $f(x) = x^n$ (where n is a positive integer), then the k-th derivative of f is given by:

$$f^{(k)}(x) = \begin{cases} 0 & \text{if } k > n \\ \frac{n!}{(n-k)!} x^{n-k} & \text{if } k \leq n \end{cases}$$

Example

Show that if A, B, and k are constants, then the funtion $y = A\cos(kt) + B\sin(kt)$ is a solution of the differential equation

$$y'' + k^2y = 0.$$

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Solution

We first compute the first and second derivatives of y:

$$y' = -Ak\sin(kt) + Bk\cos(kt),$$

$$y'' = -Ak^2\cos(kt) - Bk^2\sin(kt).$$

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Now, substituting y and y'' into the differential equation:

$$y'' + k^2 y = (-Ak^2\cos(kt) - Bk^2\sin(kt)) + k^2(A\cos(kt) + B\sin(kt)) = 0.$$

Thus, the function satisfies the differential equation.

Example

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Solution

We can use the formula for the n-th derivative of a function of the form $y = (1+x)^{-1}$:

$$y^{(n)}(x) = (-1)^n \frac{n!}{(1+x)^{n+1}}.$$

This can be derived using the power rule.

Using Differentials and Derivatives

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This represents the change in y for a small change in x, denoted dx.

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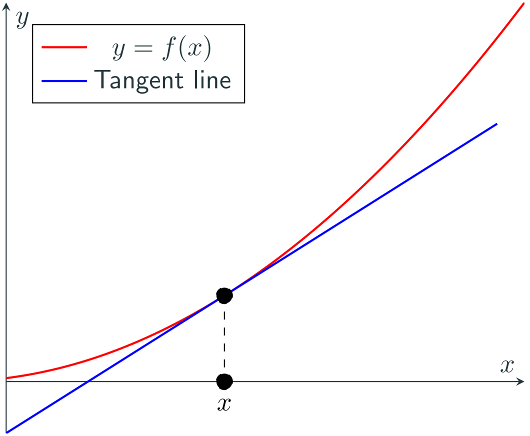

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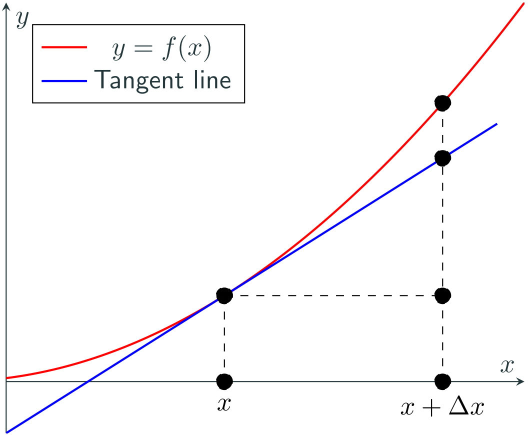

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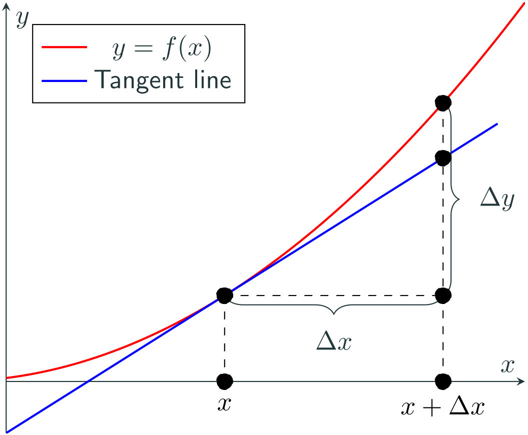

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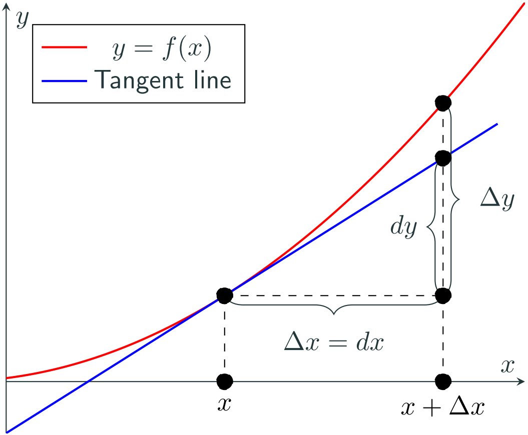

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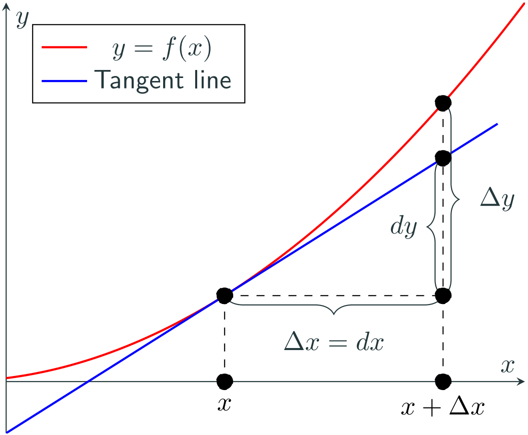

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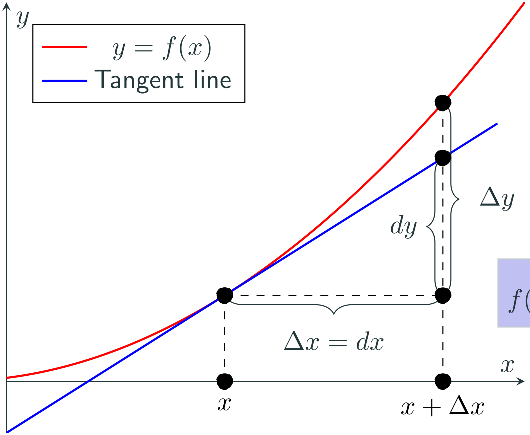

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$$f(x+\Delta x) \approx f(x) + f'(x)dx$$

Using Differentials and Derivatives

Example

Without using a scientific calculator, determine by approximately how much the value of $\sin x$ increases as x increases from $\pi/3$ to $\pi/3 + 0.006$. To 3 decimal places, what is the value of $\sin(\pi/3 + 0.006)$?

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Solution

We can use the differential to approximate the change in $\sin x$*:*

$$dy = \cos\left(\frac{\pi}{3}\right) dx = \frac{1}{2} \cdot 0.006 = 0.003.$$

Thus, the increase in $\sin x$ *is approximately* 0.003*.*

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$$dy = \cos\left(\frac{\pi}{3}\right) dx = \frac{1}{2} \cdot 0.006 = 0.003.$$

Thus, the increase in $\sin x$ *is approximately* 0.003*.*

Now, we can find the value of $\sin\left(\frac{\pi}{3} + 0.006\right)$*:*

$$\sin\left(\frac{\pi}{3} + 0.006\right) \approx \sin\left(\frac{\pi}{3}\right) + dy = \frac{\sqrt{3}}{2} + 0.003.$$

To 3 decimal places, this is approximately $0.866 + 0.003 = 0.869$*.*

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Solution

• *We can use the formula for the area of a circle,* $A = \pi r^2$*, and differentiate it with respect to* r*:* $dA = 2\pi r\, dr$*.*

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• *The relative change in area is then:* $\dfrac{dA}{A} = \dfrac{2\pi r\, dr}{\pi r^2} = \dfrac{2\, dr}{r}$.

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Solution

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- *The relative change in area is then:* $\dfrac{dA}{A} = \dfrac{2\pi r\, dr}{\pi r^2} = \dfrac{2\, dr}{r}$.
- *If* $dr = 0.02r$ *(a 2% increase), then:* $\dfrac{dA}{A} = \dfrac{2 \cdot 0.02r}{r} = 0.04$.

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- *The relative change in area is then:* $\frac{dA}{A} = \frac{2\pi r\, dr}{\pi r^2} = \frac{2\, dr}{r}$.
- *If* $dr = 0.02r$ *(a 2% increase), then:* $\frac{dA}{A} = \frac{2 \cdot 0.02r}{r} = 0.04$.
- *Thus, the percentage change in area is approximately 4%.*

Average and Instantaneous Rates of Change

Definition.

The average rate of change of a function f over an interval $[a, b]$ is given by:

$$\text{Average Rate of Change} = \frac{f(b) - f(a)}{b - a}.$$

The instantaneous rate of change at a point a is given by the derivative:

$$\text{Instantaneous Rate of Change} = f'(a).$$

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Solution

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$$\frac{dA}{dr} = 2\pi r.$$

When the radius is $r = 5$ m, the instantaneous rate of change of the area with respect to the radius is:

$$\left.\frac{dA}{dr}\right|_{r=5} = 2\pi(5) = 10\pi \ m^2/m.$$

Example

How fast is area A of a circle increasing with respect to its radius when the radius is 5 m?

Solution

We can use the formula for the area of a circle, $A = \pi r^2$, and differentiate it with respect to r:

$$\frac{dA}{dr} = 2\pi r.$$

When the radius is $r = 5$ m, the instantaneous rate of change of the area with respect to the radius is:

$$\left.\frac{dA}{dr}\right|_{r=5} = 2\pi(5) = 10\pi \ m^2/m.$$

This means that a small change Δr m in the radius when the radius is 5 m would result in a change of about $10\pi\Delta r$ m^2 in the area of the circle.