MAT123 MATHEMATICS I

Lecture 10: The Mean Value Theorem, Implicit Differentiation

Outline

The Mean Value Theorem

- Mathematical Consequences of the Mean Value Theorem
- Rolle's Theorem
- Increasing and Decreasing Functions

Implicit Differentiation

The Mean Value Theorem

The Mean Value Theorem

Theorem *(The Mean Value Theorem)*

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

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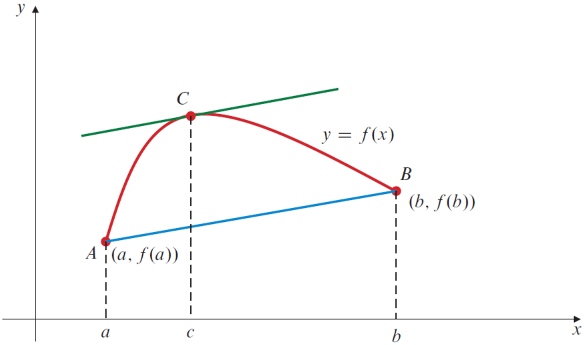

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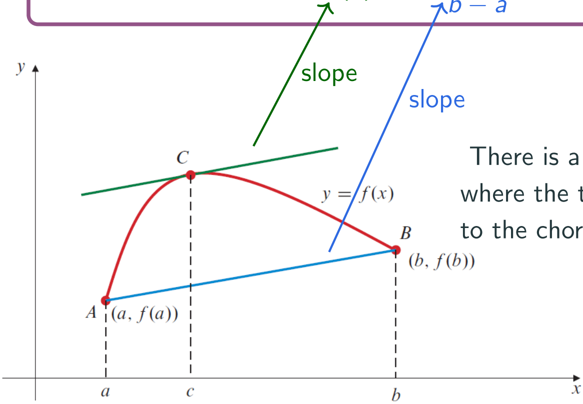
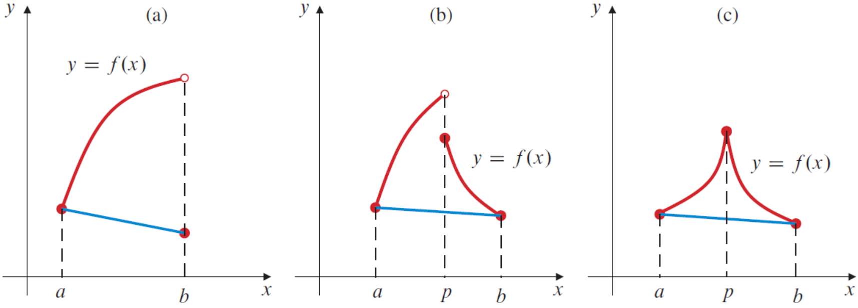


There is a point C on the curve where the tangent (green) is parallel to the chord AB (blue)

The Mean Value Theorem



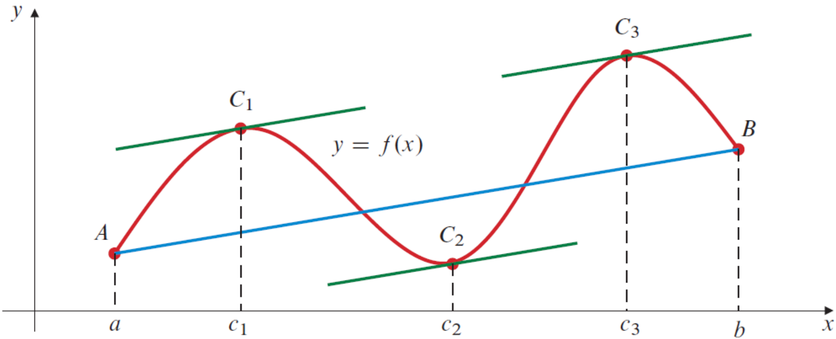
Functions that fail to satisfy the hypotheses of the Mean-Value Theorem and for which the conclusion is false:

(a) f is discontinuous at endpoint b

(b) f is discontinuous at p

(c) f is not differentiable at p

The Mean Value Theorem



The Mean Value Theorem guarantees at least one point c in (a, b) where the tangent is parallel to the secant line, but there can be multiple such points.

The Mean Value Theorem

Example

Show that $\sin x < x$ for all $x > 0$.

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Proof.

- If $x \geq \pi/2$, then since $\sin x \leq 1$, we have $\sin x < \pi/2 \leq x$.

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Proof.

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• Assuming $0 < x < \pi/2$, consider the function $f(x) = \sin x$ on the interval $[0, x]$ for $x > 0$. The function is continuous and differentiable on this interval. By the Mean Value Theorem, there exists a point $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{\sin x - 0}{x} = \frac{\sin x}{x}.$$

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• Since $f'(x) = \cos x$, we have $\cos c = \dfrac{\sin x}{x}$.

• For $c \in (0, x)$, we know that $0 < c < \pi/2$, hence $0 < \cos c < 1$.

Therefore,

$$0 < \cos c = \frac{\sin x}{x} < 1 \Rightarrow \sin x < x \text{ for } x \in (0, \pi/2).$$

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- So $\sin x < x$ for all $x > 0$. □

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• Consider the function $f(x) = \sqrt{1+x}$ on the interval $[0, x]$ for $x > 0$ and $[x, 0]$ for $-1 \leq x < 0$. The function is continuous on these intervals and differentiable on their interiors. By the MVT, there exists a point $c \in (x, 0)$ for $-1 \leq x < 0$ and $d \in (0, x)$ for $x > 0$ such that $f'(c) = f'(d) = \dfrac{f(x) - f(0)}{x - 0} = \dfrac{\sqrt{1+x} - 1}{x}$.

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$f'(c) = f'(d) = \dfrac{f(x) - f(0)}{x - 0} = \dfrac{\sqrt{1+x} - 1}{x}$.

• Since $f'(x) = \dfrac{1}{2\sqrt{1+x}}$, we have $\dfrac{1}{2\sqrt{1+c}} = \dfrac{\sqrt{1+x} - 1}{x} = \dfrac{1}{2\sqrt{1+d}}$.

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• For $d \in (0, x)$,

$$0 < d < x \implies 1 < \sqrt{1+d} \implies 2 < 2\sqrt{1+d}$$
$$\implies \frac{1}{2} > \frac{1}{2\sqrt{1+d}} = \frac{\sqrt{1+x}-1}{x} \implies \sqrt{1+x} < 1 + \frac{x}{2}.$$

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• For $c \in (x, 0)$ with $-1 \leq x < 0$,

$$-1 < c < 0 \implies 0 < 2\sqrt{1+c} < 2 \implies \frac{1}{2\sqrt{1+c}} > \frac{1}{2}$$
$$\implies \frac{\sqrt{1+x} - 1}{x} > \frac{1}{2} \overset{x<0}{\implies} \sqrt{1+x} < 1 + \frac{x}{2}.$$

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The Mean Value Theorem

Mathematical Consequences

- If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

The Mean Value Theorem

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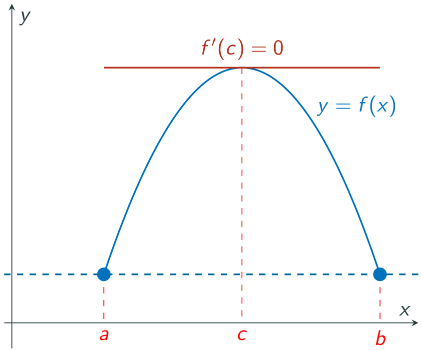
- If $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

The Mean Value Theorem

Rolle's Theorem

Suppose that $y = f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and that $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.



The Mean Value Theorem

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• *Let* $f(x) = x^3 + 3x + 1$. *Then* $f(0) = 1 > 0$ *and* $f(-1) = -3 < 0$, *so by the Intermediate Value Theorem, there exists a point* $c \in (-1, 0)$ *such that* $f(c) = 0$.

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• *However, we have* $f'(x) = 3x^2 + 3$. *Since* $f'(x) > 0$ *for all* x, *this contradicts the existence of such a point* c.

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• *Therefore, there is exactly one solution to the equation* $f(x) = 0$.

The Mean Value Theorem

Increasing and Decreasing Functions

Suppose that the function f is defined on an interval I.

• If $f(x_1) < f(x_2)$ for all x_1 and x_2 in I with $x_1 < x_2$, then f is **increasing** on I.

The Mean Value Theorem

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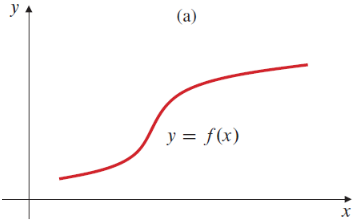
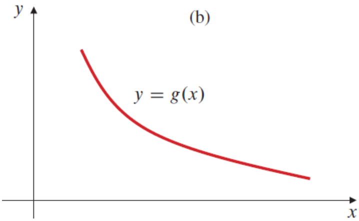
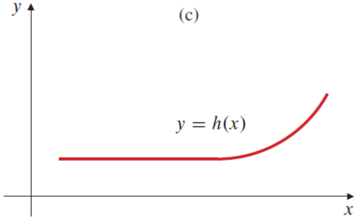


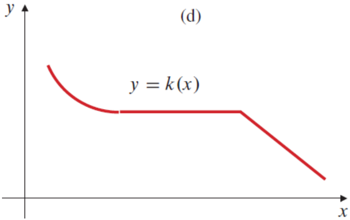
Function f is increasing.



Function g is decreasing.



Function h is nondecreasing.



Function k is nonincreasing.

The Mean Value Theorem

Theorem.

Let J be an open interval, and let I be an interval consisting of all the points in J and one or both of the endpoints of J. Suppose that f is continuous on I and differentiable on J.

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Let J be an open interval, and let I be an interval consisting of all the points in J and one or both of the endpoints of J. Suppose that f is continuous on I and differentiable on J.

- If $f'(x) > 0$ for all x in J, then f is increasing on I.

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The Mean Value Theorem

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• *Next, we set the derivative equal to zero to find the critical points:*

$$3x^2 - 12 = 0 \implies x^2 = 4 \implies x = -2, 2.$$

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• *For* $x > 2$, $f'(x) > 0$, *hence* f *is increasing on* $(2, \infty)$.

The Mean Value Theorem

Example

Show that $f(x) = x^3$ is increasing on any interval.

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• *To show that* $f(x) = x^3$ *is increasing on any interval, we can find its derivative:* $f'(x) = 3x^2$.

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Example

Show that $f(x) = x^3$ is increasing on any interval.

Solution

• *To show that $f(x) = x^3$ is increasing on any interval, we can find its derivative: $f'(x) = 3x^2$.*

• *Since $3x^2 \geq 0$ for all x, the derivative is nonnegative everywhere. This means that f is nondecreasing on any interval.*

• *Furthermore, $f'(x) = 0$ only at $x = 0$, and $f'(x) > 0$ for all $x \neq 0$. Therefore, f is strictly increasing on any interval.*

Implicit Differentiation

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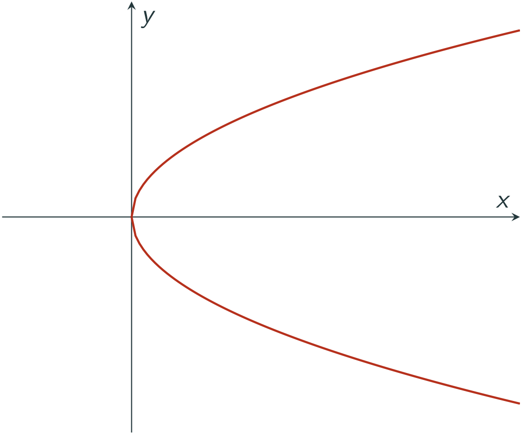

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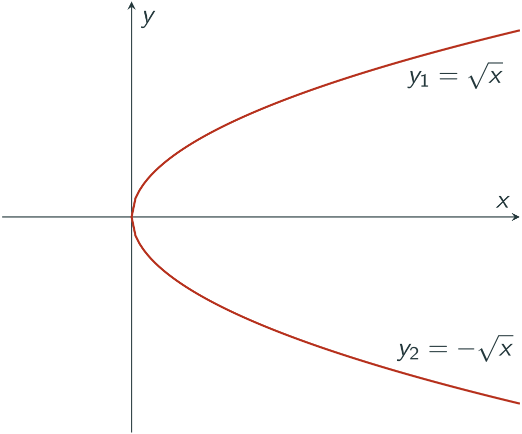

The equation $y^2 = x$ defines two differentiable functions of x on the interval $x \geq 0$.

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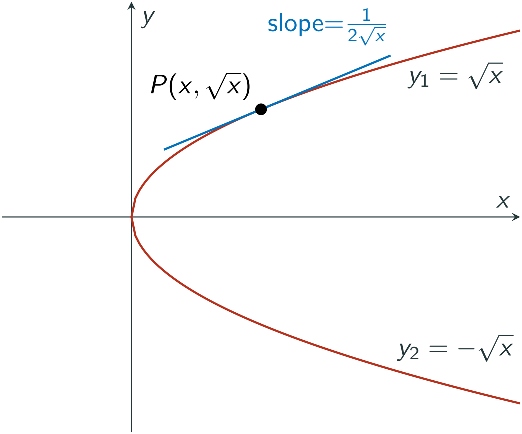
- For example, consider the equation $y^2 - x = 0$. This defines a curve in the xy-plane, which is the union of graphs of the functions $y = \pm\sqrt{x}$.



The point $P(x, \sqrt{x})$ lies on the curve $y = \sqrt{x}$, and the tangent line at this point has slope $\frac{dy}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$.

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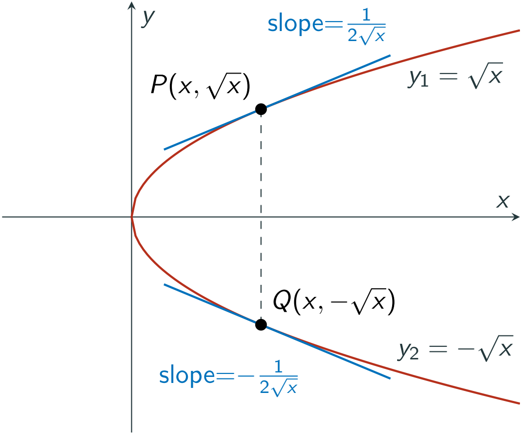

Similarly, the point $Q(x, -\sqrt{x})$ lies on the curve $y = -\sqrt{x}$, and the tangent line at this point has slope $\frac{dy}{dx}(-\sqrt{x}) = -\frac{1}{2\sqrt{x}}$.

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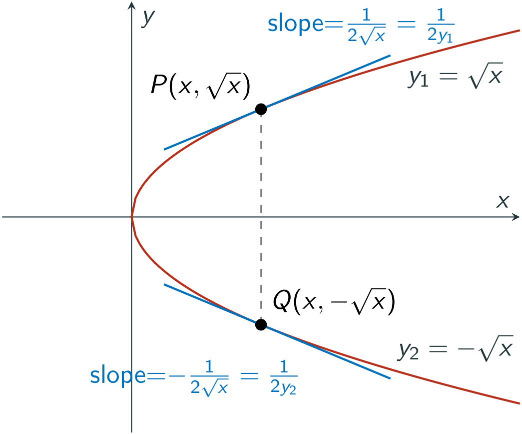

Differentiating implicitly, we treat y as a function of x and apply the chain rule:
$\frac{d}{dx}(y^2) = \frac{d}{dx}(x) \implies$
$2y\frac{dy}{dx} = 1 \implies$
$\frac{dy}{dx} = \frac{1}{2y}$.

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• *Now, we can substitute the coordinates of the point* $(3, -4)$*:*

$$\frac{dy}{dx} = -\frac{3}{-4} = \frac{3}{4}$$

Therefore, the slope of the circle at the point $(3, -4)$ *is* $\frac{3}{4}$.

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- *Finally, solving for $\frac{dy}{dx}$:*

$$\frac{dy}{dx} = \frac{3x^2 - y \cos x}{\sin x + \sin y}$$