

MAT123 MATHEMATICS I

Lecture 11: Implicit Differentiation, Applications of
Differentiation

Outline

Implicit Differentiation

- Derivatives of Inverse Functions

- Derivatives of Logarithmic and Exponential Functions

- Logarithmic Differentiation

- Derivatives of Inverse Trigonometric Functions

Applications of Differentiation

- Related Rates Problems

Implicit Differentiation

Derivatives of Inverse Functions

- Suppose that the function f is differentiable on an interval (a, b) and that either $f'(x) > 0$ for $a < x < b$, so that f is increasing on (a, b) , or $f'(x) < 0$ for $a < x < b$, so that f is decreasing on (a, b) .

Derivatives of Inverse Functions

- Suppose that the function f is differentiable on an interval (a, b) and that either $f'(x) > 0$ for $a < x < b$, so that f is increasing on (a, b) , or $f'(x) < 0$ for $a < x < b$, so that f is decreasing on (a, b) .
- In either case, f is one-to-one on (a, b) , and the inverse function f^{-1} exists.

Derivatives of Inverse Functions

- Suppose that the function f is differentiable on an interval (a, b) and that either $f'(x) > 0$ for $a < x < b$, so that f is increasing on (a, b) , or $f'(x) < 0$ for $a < x < b$, so that f is decreasing on (a, b) .
- In either case, f is one-to-one on (a, b) , and the inverse function f^{-1} exists.
- Let $y = f^{-1}$. We want to find dy/dx .

Derivatives of Inverse Functions

- Suppose that the function f is differentiable on an interval (a, b) and that either $f'(x) > 0$ for $a < x < b$, so that f is increasing on (a, b) , or $f'(x) < 0$ for $a < x < b$, so that f is decreasing on (a, b) .
- In either case, f is one-to-one on (a, b) , and the inverse function f^{-1} exists.
- Let $y = f^{-1}$. We want to find dy/dx .
- We solve the equation $y = f^{-1}(x)$ for x in terms of y : $x = f(y)$.

Derivatives of Inverse Functions

- Suppose that the function f is differentiable on an interval (a, b) and that either $f'(x) > 0$ for $a < x < b$, so that f is increasing on (a, b) , or $f'(x) < 0$ for $a < x < b$, so that f is decreasing on (a, b) .
- In either case, f is one-to-one on (a, b) , and the inverse function f^{-1} exists.
- Let $y = f^{-1}$. We want to find dy/dx .
- We solve the equation $y = f^{-1}(x)$ for x in terms of y : $x = f(y)$.
- Differentiating both sides with respect to x gives:

$$1 = f'(y) \frac{dy}{dx}$$

Derivatives of Inverse Functions

- Suppose that the function f is differentiable on an interval (a, b) and that either $f'(x) > 0$ for $a < x < b$, so that f is increasing on (a, b) , or $f'(x) < 0$ for $a < x < b$, so that f is decreasing on (a, b) .
- In either case, f is one-to-one on (a, b) , and the inverse function f^{-1} exists.
- Let $y = f^{-1}$. We want to find dy/dx .
- We solve the equation $y = f^{-1}(x)$ for x in terms of y : $x = f(y)$.
- Differentiating both sides with respect to x gives:

$$1 = f'(y) \frac{dy}{dx}$$

- Rearranging gives:

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Derivatives of Inverse Functions

Example

Show that $f(x) = x^3 + x$ is one-to-one on the whole real line, and, noting that $f(2) = 10$, find $(f^{-1})'(10)$.

Derivatives of Inverse Functions

Example

Show that $f(x) = x^3 + x$ is one-to-one on the whole real line, and, noting that $f(2) = 10$, find $(f^{-1})'(10)$.

Solution

• Since $f'(x) = 3x^2 + 1 > 0$ for all x , the function is strictly increasing and therefore one-to-one.

Derivatives of Inverse Functions

Example

Show that $f(x) = x^3 + x$ is one-to-one on the whole real line, and, noting that $f(2) = 10$, find $(f^{-1})'(10)$.

Solution

- Since $f'(x) = 3x^2 + 1 > 0$ for all x , the function is strictly increasing and therefore one-to-one.
- Now, $f(2) = 10$, gives $f^{-1}(10) = 2$, and so

$$\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(2)} = \frac{1}{3(2^2) + 1} = \frac{1}{13}$$

Derivatives of Logarithmic and Exponential Functions

Theorem.

- If $y = \log_a x$, then $\frac{dy}{dx} = \frac{1}{x \ln a}$.

Derivatives of Logarithmic and Exponential Functions

Theorem.

- If $y = \log_a x$, then $\frac{dy}{dx} = \frac{1}{x \ln a}$.
- In particular, $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Derivatives of Logarithmic and Exponential Functions

Theorem.

- If $y = \log_a x$, then $\frac{dy}{dx} = \frac{1}{x \ln a}$.
- In particular, $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Given a positive real number $a \neq 1$,

$$y = a^x \implies \log_a y = x \implies 1 = \frac{1}{(\ln a)y} \cdot y' \implies y' = y \ln a = a^x \ln a.$$

Derivatives of Logarithmic and Exponential Functions

Example

Find the derivatives of **(a)** $\ln |\cos x|$ and **(b)** $\ln (x + \sqrt{x^2 + 1})$.

Derivatives of Logarithmic and Exponential Functions

Example

Find the derivatives of **(a)** $\ln |\cos x|$ and **(b)** $\ln (x + \sqrt{x^2 + 1})$.

Solution

- For part (a), we have:

$$\frac{d}{dx}(\ln |\cos x|) = \frac{1}{|\cos x|} \cdot \frac{|\cos x|}{\cos x} \cdot (-\sin x) = -\tan x.$$

Derivatives of Logarithmic and Exponential Functions

Example

Find the derivatives of **(a)** $\ln |\cos x|$ and **(b)** $\ln (x + \sqrt{x^2 + 1})$.

Solution

- For part (a), we have:

$$\frac{d}{dx}(\ln |\cos x|) = \frac{1}{|\cos x|} \cdot \frac{|\cos x|}{\cos x} \cdot (-\sin x) = -\tan x.$$

- For part (b), we use the chain rule:

$$\frac{d}{dx} \left(\ln (x + \sqrt{x^2 + 1}) \right) = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{x}{\sqrt{x^2 + 1}} \right) = \frac{1}{\sqrt{x^2 + 1}}.$$

Implicit Differentiation

Logarithmic Differentiation

Example

Find y' if $y = x^x$ for $x > 0$.

Implicit Differentiation

Logarithmic Differentiation

Example

Find y' if $y = x^x$ for $x > 0$.

Solution

- *Taking the natural logarithm of both sides, we have:*

$$\ln y = x \ln x.$$

Implicit Differentiation

Logarithmic Differentiation

Example

Find y' if $y = x^x$ for $x > 0$.

Solution

- Taking the natural logarithm of both sides, we have:

$$\ln y = x \ln x.$$

- Differentiating both sides with respect to x :

$$\frac{1}{y} \cdot y' = \ln x + 1.$$

Logarithmic Differentiation

Example

Find y' if $y = x^x$ for $x > 0$.

Solution

- Taking the natural logarithm of both sides, we have:

$$\ln y = x \ln x.$$

- Differentiating both sides with respect to x :

$$\frac{1}{y} \cdot y' = \ln x + 1.$$

- Rearranging gives:

$$y' = y(\ln x + 1) = x^x(\ln x + 1).$$

Logarithmic Differentiation

Example

Find y' if $y = (\sin t)^{\ln t}$, where $0 < t < \pi$.

Logarithmic Differentiation

Example

Find y' if $y = (\sin t)^{\ln t}$, where $0 < t < \pi$.

Solution

- Taking the natural logarithm of both sides: $\ln y = \ln(\sin t) \cdot \ln t$.

Logarithmic Differentiation

Example

Find y' if $y = (\sin t)^{\ln t}$, where $0 < t < \pi$.

Solution

- Taking the natural logarithm of both sides: $\ln y = \ln(\sin t) \cdot \ln t$.
- Differentiating both sides with respect to t :

$$\frac{1}{y} \cdot y' = \frac{\cos t}{\sin t} \cdot \ln t + \frac{\ln(\sin t)}{t}.$$

Logarithmic Differentiation

Example

Find y' if $y = (\sin t)^{\ln t}$, where $0 < t < \pi$.

Solution

- Taking the natural logarithm of both sides: $\ln y = \ln(\sin t) \cdot \ln t$.
- Differentiating both sides with respect to t :

$$\frac{1}{y} \cdot y' = \frac{\cos t}{\sin t} \cdot \ln t + \frac{\ln(\sin t)}{t}.$$

- Rearranging gives:

$$y' = y \left(\frac{\cos t}{\sin t} \cdot \ln t + \frac{\ln(\sin t)}{t} \right) = (\sin t)^{\ln t} \left(\frac{\cos t}{\sin t} \cdot \ln t + \frac{\ln(\sin t)}{t} \right).$$

Implicit Differentiation

Logarithmic Differentiation

Example

Differentiate $y = [(x + 1)(x + 2)(x + 3)]/(x + 4)$

Solution

- *We can use logarithmic differentiation:*

$$\ln y = \ln \left(\frac{(x + 1)(x + 2)(x + 3)}{(x + 4)} \right) = \ln(x+1) + \ln(x+2) + \ln(x+3) - \ln(x+4).$$

Implicit Differentiation

Logarithmic Differentiation

Example

Differentiate $y = [(x + 1)(x + 2)(x + 3)]/(x + 4)$

Solution

- *We can use logarithmic differentiation:*

$$\ln y = \ln \left(\frac{(x + 1)(x + 2)(x + 3)}{(x + 4)} \right) = \ln(x + 1) + \ln(x + 2) + \ln(x + 3) - \ln(x + 4).$$

- *Differentiating both sides with respect to x :*

$$\frac{1}{y} \cdot y' = \frac{1}{x + 1} + \frac{1}{x + 2} + \frac{1}{x + 3} - \frac{1}{x + 4}.$$

Implicit Differentiation

Logarithmic Differentiation

Example

Differentiate $y = [(x + 1)(x + 2)(x + 3)]/(x + 4)$

Solution

- *We can use logarithmic differentiation:*

$$\ln y = \ln \left(\frac{(x + 1)(x + 2)(x + 3)}{(x + 4)} \right) = \ln(x + 1) + \ln(x + 2) + \ln(x + 3) - \ln(x + 4).$$

- *Differentiating both sides with respect to x :*

$$\frac{1}{y} \cdot y' = \frac{1}{x + 1} + \frac{1}{x + 2} + \frac{1}{x + 3} - \frac{1}{x + 4}.$$

$$\begin{aligned} \Rightarrow y' &= y \left(\frac{1}{x + 1} + \frac{1}{x + 2} + \frac{1}{x + 3} - \frac{1}{x + 4} \right) \\ &= \frac{(x + 1)(x + 2)(x + 3)}{(x + 4)} \left(\frac{1}{x + 1} + \frac{1}{x + 2} + \frac{1}{x + 3} - \frac{1}{x + 4} \right). \end{aligned}$$

Implicit Differentiation

Logarithmic Differentiation

Example

Find $\left. \frac{du}{dx} \right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Implicit Differentiation

Logarithmic Differentiation

Example

Find $\left. \frac{du}{dx} \right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Solution

- Taking the natural logarithm of both sides:

$$\ln u = \frac{1}{2} \ln ((x+1)(x^2+1)(x^3+1)) = \frac{1}{2} (\ln(x+1) + \ln(x^2+1) + \ln(x^3+1)).$$

Implicit Differentiation

Logarithmic Differentiation

Example

Find $\left. \frac{du}{dx} \right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Solution

- Taking the natural logarithm of both sides:

$$\ln u = \frac{1}{2} \ln ((x+1)(x^2+1)(x^3+1)) = \frac{1}{2} (\ln(x+1) + \ln(x^2+1) + \ln(x^3+1)).$$

- Differentiating both sides with respect to x :

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{2} \left(\frac{1}{x+1} + \frac{2x}{x^2+1} + \frac{3x^2}{x^3+1} \right).$$

Implicit Differentiation

Logarithmic Differentiation

Example

Find $\left. \frac{du}{dx} \right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Solution

- Taking the natural logarithm of both sides:

$$\ln u = \frac{1}{2} \ln ((x+1)(x^2+1)(x^3+1)) = \frac{1}{2} (\ln(x+1) + \ln(x^2+1) + \ln(x^3+1)).$$

- Differentiating both sides with respect to x :

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{2} \left(\frac{1}{x+1} + \frac{2x}{x^2+1} + \frac{3x^2}{x^3+1} \right).$$

- Rearranging gives:

$$\frac{du}{dx} = u \cdot \frac{1}{2} \left(\frac{1}{x+1} + \frac{2x}{x^2+1} + \frac{3x^2}{x^3+1} \right).$$

Logarithmic Differentiation

Example

Find $\left. \frac{du}{dx} \right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Solution

- *Rearranging gives:*

$$\frac{du}{dx} = u \cdot \frac{1}{2} \left(\frac{1}{x+1} + \frac{2x}{x^2+1} + \frac{3x^2}{x^3+1} \right).$$

- *Evaluating at $x = 1$:*

$$\left. \frac{du}{dx} \right|_{x=1} = \sqrt{(1+1)(1^2+1)(1^3+1)} \cdot \frac{1}{2} \left(\frac{1}{2} + 1 + \frac{3}{2} \right) = \sqrt{8} \cdot \frac{1}{2} \cdot 3 = 3\sqrt{2}.$$

Derivatives of Inverse Trigonometric Functions

- If $y = \sin^{-1} x$, then $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

Derivatives of Inverse Trigonometric Functions

- If $y = \sin^{-1} x$, then $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
- Differentiating both sides with respect to x gives:

$$1 = \cos y \cdot \frac{dy}{dx}.$$

Derivatives of Inverse Trigonometric Functions

- If $y = \sin^{-1} x$, then $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
- Differentiating both sides with respect to x gives:

$$1 = \cos y \cdot \frac{dy}{dx}.$$

- Since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, we know that $\cos y \geq 0$. Therefore,

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

Derivatives of Inverse Trigonometric Functions

- If $y = \sin^{-1} x$, then $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
- Differentiating both sides with respect to x gives:

$$1 = \cos y \cdot \frac{dy}{dx}.$$

- Since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, we know that $\cos y \geq 0$. Therefore,

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

- It follows that

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}.$$

Derivatives of Inverse Trigonometric Functions

- If $y = \sin^{-1} x$, then $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
- Differentiating both sides with respect to x gives:

$$1 = \cos y \cdot \frac{dy}{dx}.$$

- Since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, we know that $\cos y \geq 0$. Therefore,

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

- It follows that

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}.$$

- The previous argument can be repeated to show that:

- $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}.$
- $\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}.$
- $\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}.$

Derivatives of Hyperbolic Functions

- $\frac{d}{dx} \cosh x = \sinh x.$
- $\frac{d}{dx} \sinh x = \cosh x.$
- $\frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$
- $\frac{d}{dx} \coth x = -\frac{1}{\sinh^2 x} = -\operatorname{csch}^2 x.$
- $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \cdot \tanh x.$
- $\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \cdot \coth x.$

Applications of Differentiation

Applications of Differentiation

- Related rates problems
- Root finding methods
- Graphing problems
- Optimization problems
- Approximation problems
- Evaluation of limits

Related Rates Problems

Example

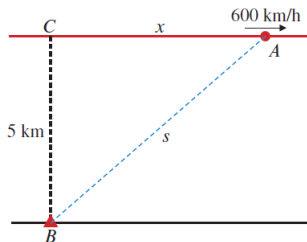
An aircraft is flying horizontally at a speed of 600 km/h. How fast is the distance between the aircraft and a radio beacon increasing 1 min after the aircraft passes 5 km directly above the beacon?

Related Rates Problems

Example

An aircraft is flying horizontally at a speed of 600 km/h. How fast is the distance between the aircraft and a radio beacon increasing 1 min after the aircraft passes 5 km directly above the beacon?

Solution



The distance s between the aircraft and the beacon is given by:

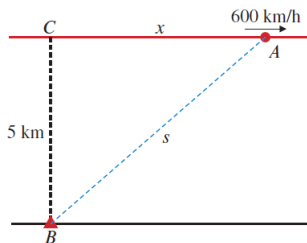
$$s^2 = x^2 + 25.$$

Related Rates Problems

Example

An aircraft is flying horizontally at a speed of 600 km/h. How fast is the distance between the aircraft and a radio beacon increasing 1 min after the aircraft passes 5 km directly above the beacon?

Solution



- We are given that $dx/dt = 600$ km/h. We want to find ds/dt when $x = 10$ km.

The distance s between the aircraft and the beacon is given by:

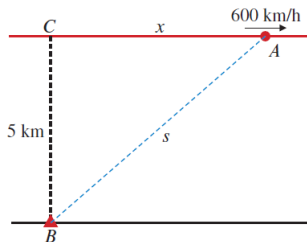
$$s^2 = x^2 + 25.$$

Related Rates Problems

Example

An aircraft is flying horizontally at a speed of 600 km/h. How fast is the distance between the aircraft and a radio beacon increasing 1 min after the aircraft passes 5 km directly above the beacon?

Solution



- We are given that $dx/dt = 600$ km/h. We want to find ds/dt when $x = 10$ km.
- Using the chain rule:

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt}. \quad (1)$$

The distance s between the aircraft and the beacon is given by:

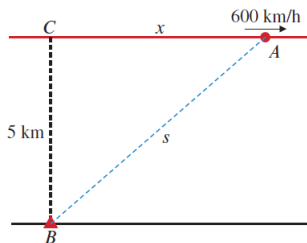
$$s^2 = x^2 + 25.$$

Related Rates Problems

Example

An aircraft is flying horizontally at a speed of 600 km/h. How fast is the distance between the aircraft and a radio beacon increasing 1 min after the aircraft passes 5 km directly above the beacon?

Solution



- We are given that $dx/dt = 600$ km/h. We want to find ds/dt when $x = 10$ km.

- Using the chain rule:

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt}. \quad (1)$$

- Since $x = 10$ km at $t = 1$ min, we have $s = \sqrt{10^2 + 5^2} = \sqrt{125} = 5\sqrt{5}$ km.

The distance s between the aircraft and the beacon is given by:

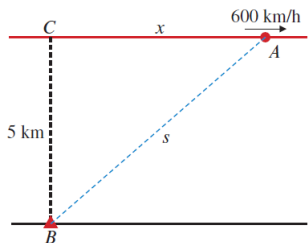
$$s^2 = x^2 + 25.$$

Related Rates Problems

Example

An aircraft is flying horizontally at a speed of 600 km/h. How fast is the distance between the aircraft and a radio beacon increasing 1 min after the aircraft passes 5 km directly above the beacon?

Solution



The distance s between the aircraft and the beacon is given by:

$$s^2 = x^2 + 25.$$

- We are given that $dx/dt = 600$ km/h. We want to find ds/dt when $x = 10$ km.
- Using the chain rule:

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt}. \quad (1)$$

- Since $x = 10$ km at $t = 1$ min, we have $s = \sqrt{10^2 + 5^2} = \sqrt{125} = 5\sqrt{5}$ km.
- Substituting the known values in (1) gives:

$$2(5\sqrt{5}) \frac{ds}{dt} = 2(10)(600) \Rightarrow \frac{ds}{dt} = \frac{1200}{\sqrt{5}} \approx 536.7 \text{ km/h.}$$

Related Rates Problems

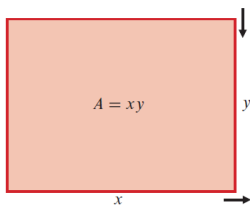
Example

How fast is the area of a rectangle changing if one side is 10 cm long and is increasing at a rate of 2 cm/s and the other side is 8 cm long and is decreasing at a rate of 3 cm/s?

Related Rates Problems

Example

How fast is the area of a rectangle changing if one side is 10 cm long and is increasing at a rate of 2 cm/s and the other side is 8 cm long and is decreasing at a rate of 3 cm/s?

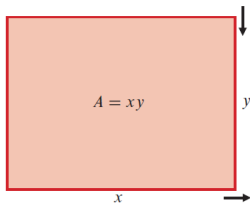


Let A be the area of the rectangle, x the length of the side that is increasing, and y the length of the side that is decreasing.

Related Rates Problems

Example

How fast is the area of a rectangle changing if one side is 10 cm long and is increasing at a rate of 2 cm/s and the other side is 8 cm long and is decreasing at a rate of 3 cm/s?



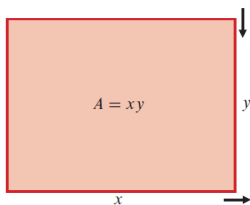
- Then: $A = xy$.

Let A be the area of the rectangle, x the length of the side that is increasing, and y the length of the side that is decreasing.

Related Rates Problems

Example

How fast is the area of a rectangle changing if one side is 10 cm long and is increasing at a rate of 2 cm/s and the other side is 8 cm long and is decreasing at a rate of 3 cm/s?



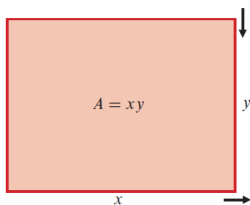
- Then: $A = xy$.
- We are given that $dx/dt = 2$ cm/s and $dy/dt = -3$ cm/s. We want to find dA/dt when $x = 10$ cm and $y = 8$ cm.

Let A be the area of the rectangle, x the length of the side that is increasing, and y the length of the side that is decreasing.

Related Rates Problems

Example

How fast is the area of a rectangle changing if one side is 10 cm long and is increasing at a rate of 2 cm/s and the other side is 8 cm long and is decreasing at a rate of 3 cm/s?



Let A be the area of the rectangle, x the length of the side that is increasing, and y the length of the side that is decreasing.

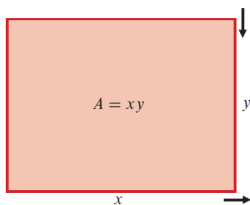
- Then: $A = xy$.
- We are given that $dx/dt = 2$ cm/s and $dy/dt = -3$ cm/s. We want to find dA/dt when $x = 10$ cm and $y = 8$ cm.
- Using the product rule:

$$\frac{dA}{dt} = x \frac{dy}{dt} + y \frac{dx}{dt}.$$

Related Rates Problems

Example

How fast is the area of a rectangle changing if one side is 10 cm long and is increasing at a rate of 2 cm/s and the other side is 8 cm long and is decreasing at a rate of 3 cm/s?



Let A be the area of the rectangle, x the length of the side that is increasing, and y the length of the side that is decreasing.

- Then: $A = xy$.
- We are given that $dx/dt = 2$ cm/s and $dy/dt = -3$ cm/s. We want to find dA/dt when $x = 10$ cm and $y = 8$ cm.
- Using the product rule:

$$\frac{dA}{dt} = x \frac{dy}{dt} + y \frac{dx}{dt}.$$

- Substituting the known values gives:

$$\frac{dA}{dt} = 10(-3) + 8(2) = -30 + 16 = -14 \text{ cm}^2/\text{s}.$$

Related Rates Problems

How to solve related-rates problems

- * Read the problem very carefully. Try to understand the relationships between the variable quantities. What is given? What is to be found?

Related Rates Problems

How to solve related-rates problems

- ✓ Read the problem very carefully. Try to understand the relationships between the variable quantities. What is given? What is to be found?
- * Make a sketch if appropriate.

Related Rates Problems

How to solve related-rates problems

- ✓ Read the problem very carefully. Try to understand the relationships between the variable quantities. What is given? What is to be found?
- ✓ Make a sketch if appropriate.
- * Define any symbols you want to use that are not defined in the statement of the problem. Express given and required quantities and rates in terms of these symbols.

Related Rates Problems

How to solve related-rates problems

- ✓ Read the problem very carefully. Try to understand the relationships between the variable quantities. What is given? What is to be found?
- ✓ Make a sketch if appropriate.
- ✓ Define any symbols you want to use that are not defined in the statement of the problem. Express given and required quantities and rates in terms of these symbols.
- ✱ From a careful reading of the problem or consideration of the sketch, identify one or more equations linking the variable quantities.

Related Rates Problems

How to solve related-rates problems

- ✓ Read the problem very carefully. Try to understand the relationships between the variable quantities. What is given? What is to be found?
- ✓ Make a sketch if appropriate.
- ✓ Define any symbols you want to use that are not defined in the statement of the problem. Express given and required quantities and rates in terms of these symbols.
- ✓ From a careful reading of the problem or consideration of the sketch, identify one or more equations linking the variable quantities.
- ✱ Differentiate the equation(s) implicitly with respect to time, regarding all variable quantities as functions of time.

Related Rates Problems

How to solve related-rates problems

- ✓ Read the problem very carefully. Try to understand the relationships between the variable quantities. What is given? What is to be found?
- ✓ Make a sketch if appropriate.
- ✓ Define any symbols you want to use that are not defined in the statement of the problem. Express given and required quantities and rates in terms of these symbols.
- ✓ From a careful reading of the problem or consideration of the sketch, identify one or more equations linking the variable quantities.
- ✓ Differentiate the equation(s) implicitly with respect to time, regarding all variable quantities as functions of time.
- * Substitute any given values for the quantities and their rates, then solve the resulting equation(s) for the unknown quantities and rates.

Related Rates Problems

How to solve related-rates problems

- ✓ Read the problem very carefully. Try to understand the relationships between the variable quantities. What is given? What is to be found?
- ✓ Make a sketch if appropriate.
- ✓ Define any symbols you want to use that are not defined in the statement of the problem. Express given and required quantities and rates in terms of these symbols.
- ✓ From a careful reading of the problem or consideration of the sketch, identify one or more equations linking the variable quantities.
- ✓ Differentiate the equation(s) implicitly with respect to time, regarding all variable quantities as functions of time.
- ✓ Substitute any given values for the quantities and their rates, then solve the resulting equation(s) for the unknown quantities and rates.
- * Make a concluding statement answering the question asked.

Related Rates Problems

Example

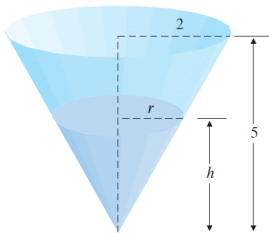
A leaky water tank is in the shape of an inverted right circular cone with depth 5 m and top radius 2 m. When the water in the tank is 4 m deep, it is leaking out at a rate of $1/12 \text{ m}^3/\text{min}$. How fast is the water level in the tank dropping at that time?

Related Rates Problems

Example

A leaky water tank is in the shape of an inverted right circular cone with depth 5 m and top radius 2 m. When the water in the tank is 4 m deep, it is leaking out at a rate of $1/12 \text{ m}^3/\text{min}$. How fast is the water level in the tank dropping at that time?

Solution



The volume V of the water in the tank is given by:

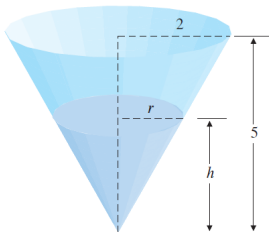
$$V = \frac{1}{3}\pi r^2 h.$$

Related Rates Problems

Example

A leaky water tank is in the shape of an inverted right circular cone with depth 5 m and top radius 2 m. When the water in the tank is 4 m deep, it is leaking out at a rate of $1/12 \text{ m}^3/\text{min}$. How fast is the water level in the tank dropping at that time?

Solution



The volume V of the water in the tank is given by:

$$V = \frac{1}{3}\pi r^2 h.$$

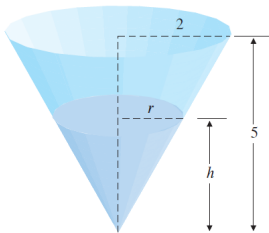
- Using similar triangles: $\frac{r}{h} = \frac{2}{5} \Rightarrow r = \frac{2}{5}h.$

Related Rates Problems

Example

A leaky water tank is in the shape of an inverted right circular cone with depth 5 m and top radius 2 m. When the water in the tank is 4 m deep, it is leaking out at a rate of $1/12 \text{ m}^3/\text{min}$. How fast is the water level in the tank dropping at that time?

Solution



The volume V of the water in the tank is given by:

$$V = \frac{1}{3}\pi r^2 h.$$

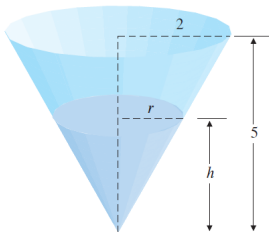
- Using similar triangles: $\frac{r}{h} = \frac{2}{5} \Rightarrow r = \frac{2}{5}h$.
- Substituting this into the volume formula gives: $V = \frac{1}{3}\pi \left(\frac{2}{5}h\right)^2 h = \frac{4\pi}{75}h^3$.

Related Rates Problems

Example

A leaky water tank is in the shape of an inverted right circular cone with depth 5 m and top radius 2 m. When the water in the tank is 4 m deep, it is leaking out at a rate of $1/12 \text{ m}^3/\text{min}$. How fast is the water level in the tank dropping at that time?

Solution



The volume V of the water in the tank is given by:

$$V = \frac{1}{3}\pi r^2 h.$$

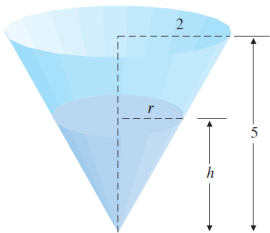
- Using similar triangles: $\frac{r}{h} = \frac{2}{5} \Rightarrow r = \frac{2}{5}h$.
- Substituting this into the volume formula gives: $V = \frac{1}{3}\pi \left(\frac{2}{5}h\right)^2 h = \frac{4\pi}{75}h^3$.
- Differentiating both sides with respect to time t gives: $\frac{dV}{dt} = \frac{4\pi}{75} \cdot 3h^2 \frac{dh}{dt} = \frac{4\pi}{25}h^2 \frac{dh}{dt}$.

Related Rates Problems

Example

A leaky water tank is in the shape of an inverted right circular cone with depth 5 m and top radius 2 m. When the water in the tank is 4 m deep, it is leaking out at a rate of $1/12 \text{ m}^3/\text{min}$. How fast is the water level in the tank dropping at that time?

Solution



The volume V of the water in the tank is given by:

$$V = \frac{1}{3}\pi r^2 h.$$

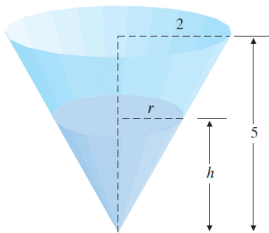
- Using similar triangles: $\frac{r}{h} = \frac{2}{5} \Rightarrow r = \frac{2}{5}h$.
- Substituting this into the volume formula gives: $V = \frac{1}{3}\pi \left(\frac{2}{5}h\right)^2 h = \frac{4\pi}{75}h^3$.
- Differentiating both sides with respect to time t gives: $\frac{dV}{dt} = \frac{4\pi}{75} \cdot 3h^2 \frac{dh}{dt} = \frac{4\pi}{25}h^2 \frac{dh}{dt}$.
- We are given that $\frac{dV}{dt} = -\frac{1}{12} \text{ m}^3/\text{min}$ when $h = 4 \text{ m}$.

Related Rates Problems

Example

A leaky water tank is in the shape of an inverted right circular cone with depth 5 m and top radius 2 m. When the water in the tank is 4 m deep, it is leaking out at a rate of $1/12 \text{ m}^3/\text{min}$. How fast is the water level in the tank dropping at that time?

Solution



The volume V of the water in the tank is given by:

$$V = \frac{1}{3}\pi r^2 h.$$

- Substituting these values gives:

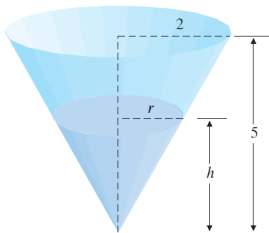
$$-\frac{1}{12} = \frac{4\pi}{25}(4^2)\frac{dh}{dt} \Rightarrow -\frac{1}{12} = \frac{64\pi}{25}\frac{dh}{dt}.$$

Related Rates Problems

Example

A leaky water tank is in the shape of an inverted right circular cone with depth 5 m and top radius 2 m. When the water in the tank is 4 m deep, it is leaking out at a rate of $1/12 \text{ m}^3/\text{min}$. How fast is the water level in the tank dropping at that time?

Solution



The volume V of the water in the tank is given by:

$$V = \frac{1}{3}\pi r^2 h.$$

- Substituting these values gives:

$$-\frac{1}{12} = \frac{4\pi}{25}(4^2) \frac{dh}{dt} \Rightarrow -\frac{1}{12} = \frac{64\pi}{25} \frac{dh}{dt}.$$

- Solving for $\frac{dh}{dt}$ gives:

$$\frac{dh}{dt} = -\frac{25}{64\pi} \cdot \frac{1}{12} = -\frac{25}{768\pi} \text{ m/min.}$$