MAT123 MATHEMATICS I

Lecture 12: Applications of Differentiation

Outline

Applications of Differentiation

Finding Roots of Equations

- One of the most important applications of differentiation is finding roots of equations.
- The basic idea is to use tangents and their x-intercepts to approximate a root.
- The method based on this idea is called Newton's method, which uses the derivative to iteratively find better approximations of the root.

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- If g is a **contraction mapping** (as in the following theorem), then the iteration converges to the unique fixed point.
- This method can be used to find roots of equations by rewriting the equation in the form $x = g(x)$.

Finding Roots of Equations

A fixed-point theorem

If g is a contraction mapping on a closed interval $I = [a, b]$, that is, if

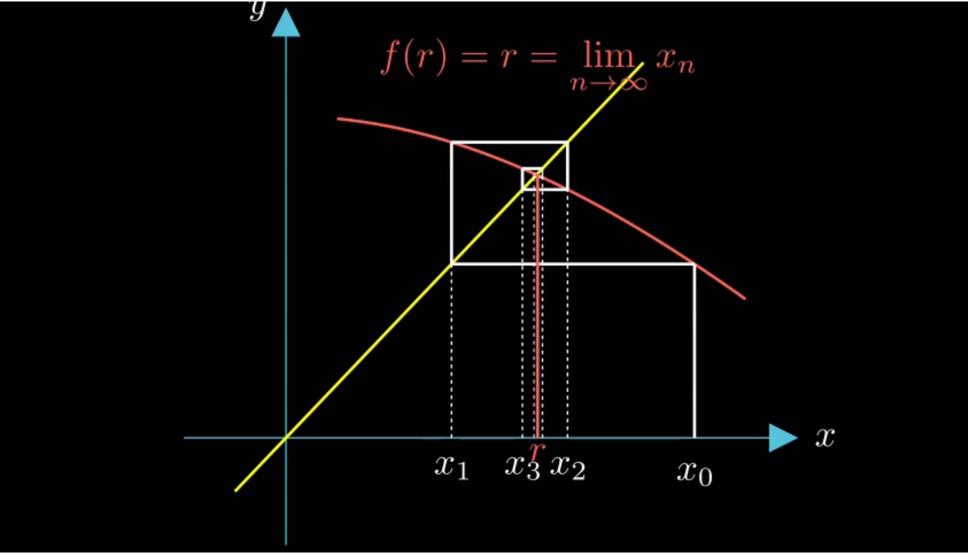
1. $g(x)$ belongs to I whenever x belongs to I, and
2. there exists a constant $0 < K < 1$ such that
   $|g(x) - g(y)| \leq K|x - y|$, for all $x, y \in [a, b]$,

then g has a unique fixed point r in I. More precisely, there exists a unique $r \in I$ such that $g(r) = r$ and starting from any number $x_0 \in I$, the iteration

$$x_1 = g(x_0), x_2 = g(x_1), \ldots, x_{n+1} = g(x_n), \ldots$$

gets closer and closer to r.

Finding Roots of Equations



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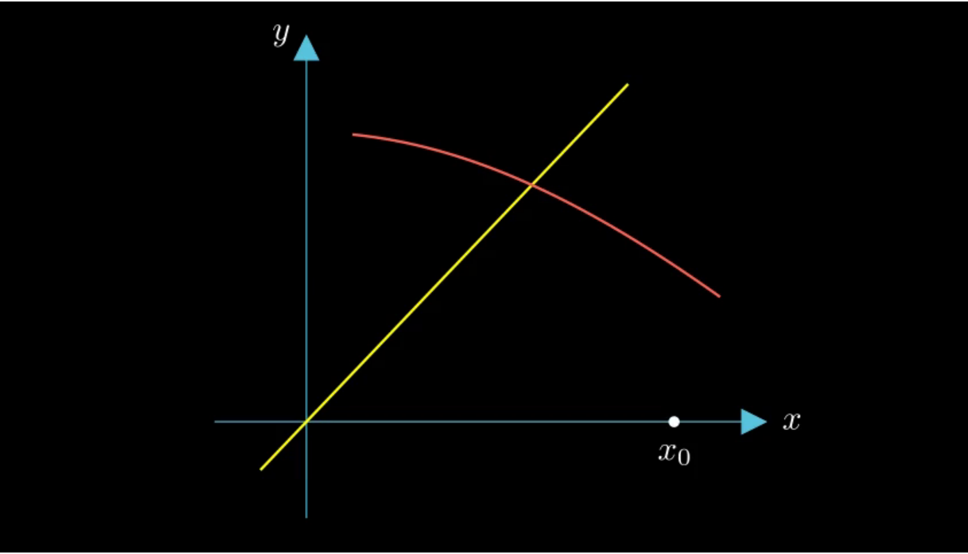
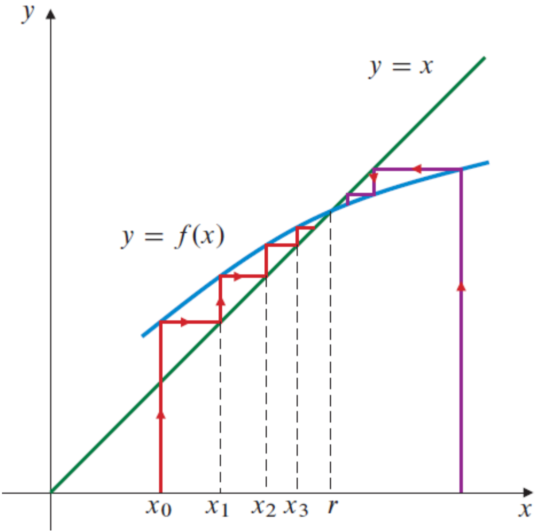


Finding Roots of Equations

A function of positive slope where the slope is less than 1 (In this case we have a "staircase" rather than a "spiral".):



Finding Roots of Equations

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$$x_1 = g(x_0), x_2 = g(x_1), \ldots, x_{n+1} = g(x_n), \ldots$$

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The values of the sequence are shown in the table on the right.

n	x_n
0	0.2
1	0.19601332
2	0.19617016
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The root is 0.19616428 to 8 decimal places.

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- And we continue this process to produce approximating numbers.

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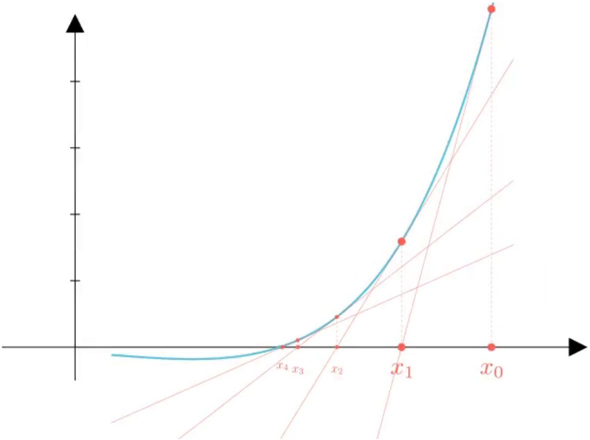

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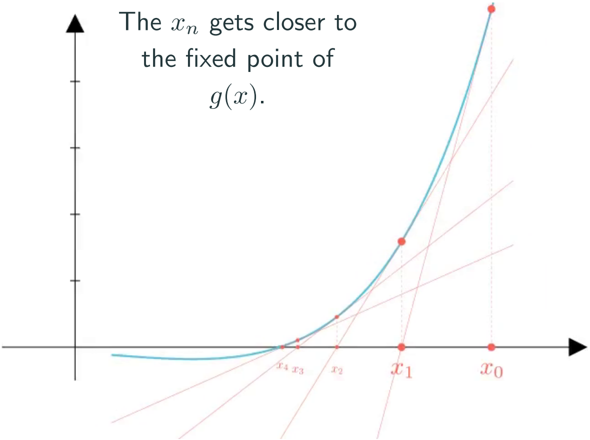

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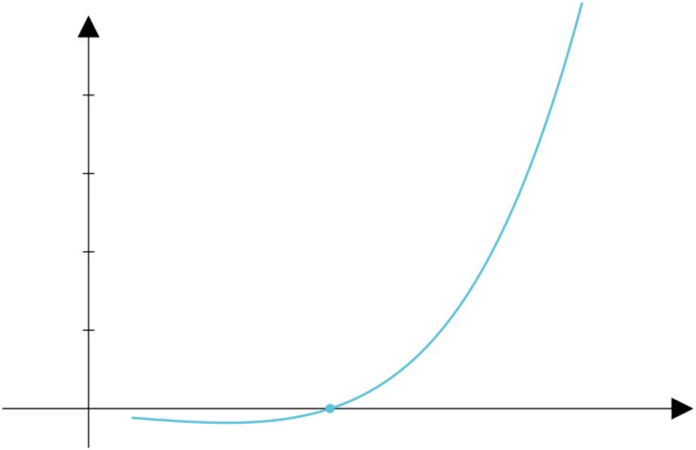

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We have $f(x) = x^3 - x - 1$ *and* $f'(x) = 3x^2 - 1$. *Since* f *is continuous and since* $f(1) = -1$ *and* $f(2) = 5$, *the equation has a root in the interval* $[1, 2]$.

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1} = \frac{2x_n^3 + 1}{3x_n^2}$$

n	x_n	$f(x_n)$
0	1.5	0.875
1	1.34782608696...	0.100682173091...
2	1.32520039895...	0.002058361917...
3	1.32471817400...	0.000000924378...
4	1.32471795724...	0.000000000000...
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  - Series expansion

Indeterminate Forms

The first l'Hospital Rule

Suppose the functions $f(x)$ and $g(x)$ are differentiable on the interval (a, b), and $g'(x) \neq 0$ there. Suppose also that

- $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$ and
- $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$, where L is finite or $\pm\infty$.

Then

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Similar results hold if every occurrence of $\lim_{x \to a^+}$ is replaced by $\lim_{x \to b^-}$ or even $\lim_{x \to c}$ where c is any point in the interval (a, b). The case $a = -\infty$ and $b = \infty$ can be handled similarly.

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We have $\left[\frac{0}{0}\right]$ *for*

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Solution

We have $\left[\frac{0}{0}\right]$ *for*

$$\lim_{x\to 1} \frac{\ln x}{x^2 - 1}.$$

Applying l'Hôpital's Rule, we differentiate the numerator and denominator:

$$\lim_{x\to 1} \frac{\ln x}{x^2 - 1} = \lim_{x\to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^2 - 1)} = \lim_{x\to 1} \frac{\frac{1}{x}}{2x} = \lim_{x\to 1} \frac{1}{2x^2} = \frac{1}{2}.$$

Example

Evaluate $\lim_{x \to 0} \dfrac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2}$.

Example

Evaluate $\lim\limits_{x\to 0}\dfrac{2\sin x-\sin(2x)}{2e^x-2-2x-x^2}$.

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- *We still have the indeterminate form* $\frac{0}{0}$. *Applying l'Hôpital's Rule again:*

$$= \lim_{x\to 0} \frac{-2\sin x + 4\sin(2x)}{2e^x - 2}$$

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$$= \lim_{x \to 0} \frac{-2\cos x + 8\cos(2x)}{2e^x} = \frac{-2+8}{2} = 3.$$

Example

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Solution

(a) *We have* $\lim_{x\to(\pi/2)^-} \frac{2x-\pi}{\cos^2 x} = \frac{0}{0}$. *Applying l'Hôpital's Rule:*

$$\begin{aligned}\lim_{x\to(\pi/2)^-} \frac{2x-\pi}{\cos^2 x} &= \lim_{x\to(\pi/2)^-} \frac{2}{-2\cos x \sin x} \\ &= \lim_{x\to(\pi/2)^-} \frac{-1}{\cos x \sin x} \\ &= -\infty.\end{aligned}$$

Example

Evaluate (a) $\lim_{x \to (\pi/2)^-} \frac{2x - \pi}{\cos^2 x}$ and (b) $\lim_{x \to 1^+} \frac{x}{\ln x}$.

Solution

(b) l'Hôpital's Rule cannot be used here since the limit is not of an indeterminate form. Indeed, the denominator approaches 0 *from the positive side as* $x \to 1^+$*, while the numerator approaches* 1*, so we have*

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Indeterminate Forms

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Had we tried to apply l'Hôpital's Rule, we would have found that

$$\lim_{x \to 1^+} \frac{x}{\ln x} = \lim_{x \to 1^+} \frac{1}{\frac{1}{x}} = \lim_{x \to 1^+} x = 1.$$

This shows that l'Hôpital's Rule is not always applicable, and we must be careful to identify the correct form before applying it.

Indeterminate Forms

Example

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$.

Solution

Indeterminate Forms

Example

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$.

Solution

- *Given limit is of the form* $\infty - \infty$. *Manipulating the expression, we have*

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- *This is of the form $\frac{0}{0}$, so we can apply l'Hôpital's Rule:*

$$= \lim_{x \to 0^+} \frac{\cos x - 1}{\sin x + x \cos x},$$

still of the form $\frac{0}{0}$.

Example

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still of the form $\frac{0}{0}$. *Applying l'Hôpital's Rule again:*

$$=\lim_{x\to 0^+}\frac{-\sin x}{\cos x+\cos x - x\sin x}=\lim_{x\to 0^+}\frac{-\sin x}{2\cos x - x\sin x}.$$

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still of the form $\frac{0}{0}$. Applying l'Hôpital's Rule again:

$$= \lim_{x \to 0^+} \frac{-\sin x}{\cos x + \cos x - x \sin x} = \lim_{x \to 0^+} \frac{-\sin x}{2 \cos x - x \sin x}.$$

- *Evaluating the limit gives $\frac{-0}{2-0} = 0$.*

Indeterminate Forms

The second l'Hospital's Rule

Suppose the functions $f(x)$ and $g(x)$ are differentiable on the interval (a, b), and $g'(x) \neq 0$ there. Suppose also that

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Suppose the functions $f(x)$ and $g(x)$ are differentiable on the interval (a, b), and $g'(x) \neq 0$ there. Suppose also that

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- $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, where L is finite or $\pm\infty$.

Then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L.$$

Similar results hold if every occurrence of $\lim_{x\to a^+}$ is replaced by $\lim_{x\to b^-}$ or even $\lim_{x\to c}$ where c is any point in the interval (a, b). The case $a = -\infty$ and $b = \infty$ can be handled similarly.

Example

Evaluate (a) $\lim_{x\to\infty} \frac{x^2}{e^x}$ and (b) $\lim_{x\to 0^+} x^a \ln x$, where $a > 0$.

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Solution

(a) We have $\lim_{x\to\infty} \frac{x^2}{e^x} = \frac{\infty}{\infty}$*. Applying l'Hôpital's Rule twice:*

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

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(b) We have $\lim_{x\to 0^+} x^a \ln x = 0 \cdot (-\infty)$*. Rewriting gives:*

$$= \lim_{x\to 0^+} \frac{\ln x}{1/x^a},$$

which is of the form $\frac{-\infty}{\infty}$*, so we can apply l'Hôpital's Rule:*

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$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{a}{x^{a+1}}} = \lim_{x\to 0^+} \frac{x^a}{-a} = 0.$$

Indeterminate Forms

Example

Evaluate $\lim_{x \to 0^+} x^x$.

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Therefore, $\lim_{x\to 0^+} x^x = e^0 = 1.$

Indeterminate Forms

Example

Evaluate $\lim_{x\to\infty}\left(1+\sin\dfrac{3}{x}\right)^{x}$.

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We have

$$\lim_{x\to\infty}\left(1+\sin\frac{3}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\sin\frac{3}{x}\right)}.$$

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This is of the form $\frac{0}{0}$*, so we can apply l'Hôpital's Rule:*

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\sin\frac{3}{x}}\cdot\cos\frac{3}{x}\cdot\left(-\frac{3}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{\cos\frac{3}{x}}{1+\sin\frac{3}{x}}\cdot 3 = 3.$$

Example

Evaluate $\lim_{x\to\infty}\left(1+\sin\frac{3}{x}\right)^x = e^3.$

Solution

We have

$$\lim_{x\to\infty}\left(1+\sin\frac{3}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\sin\frac{3}{x}\right)}.$$

Now we need to evaluate $\lim_{x\to\infty} x\ln\left(1+\sin\frac{3}{x}\right)$, *which is of the form* $\infty\cdot 0$. *We can rewrite it as*

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Extreme Values

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Absolute extreme values

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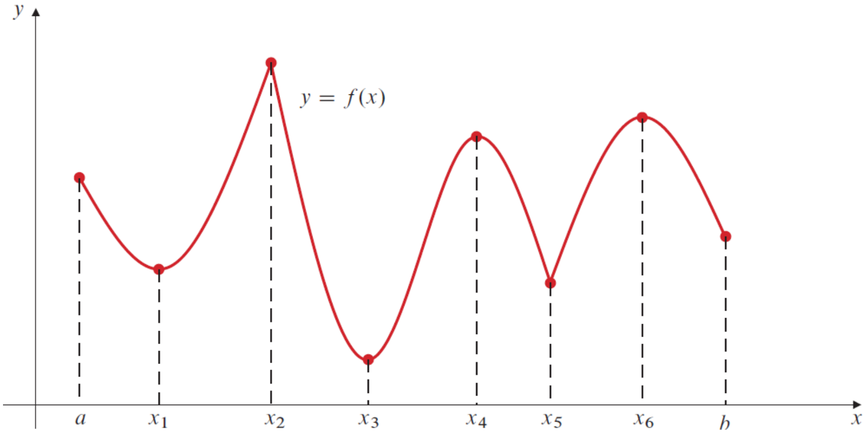
Theorem *(Existence of extreme values)*

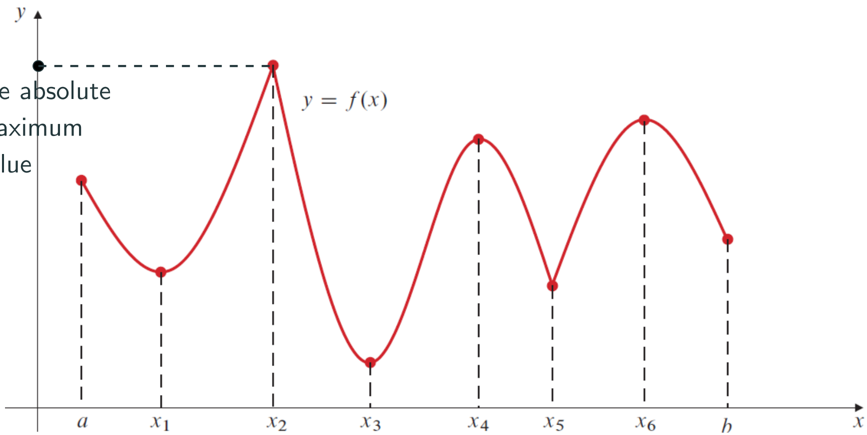
If f is continuous on a finite closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum value on $[a, b]$.

Extreme Values

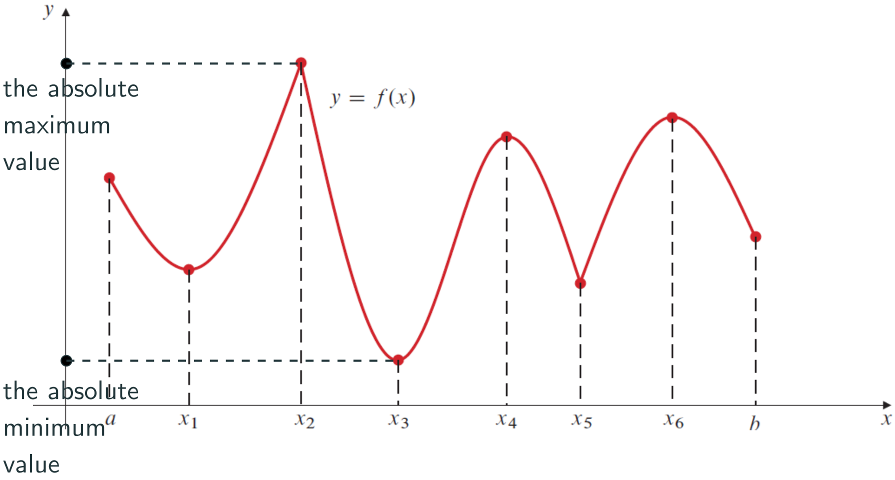


Extreme Values

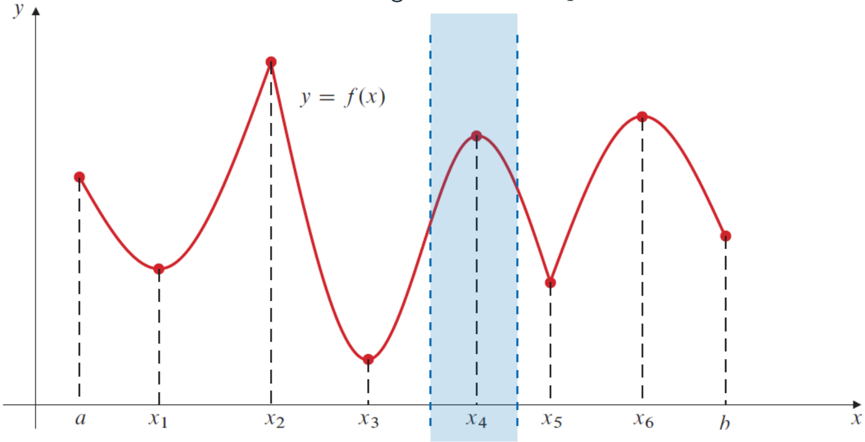
the absolute
maximum
value



Extreme Values

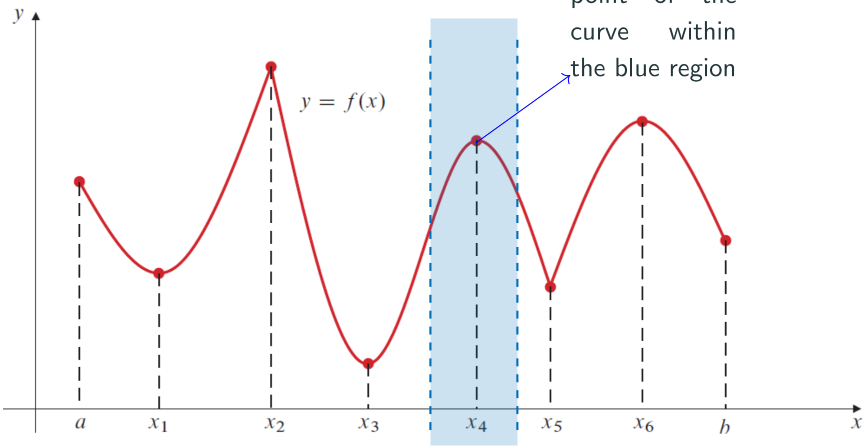


Let's restrict f to a neighborhood of x_4.



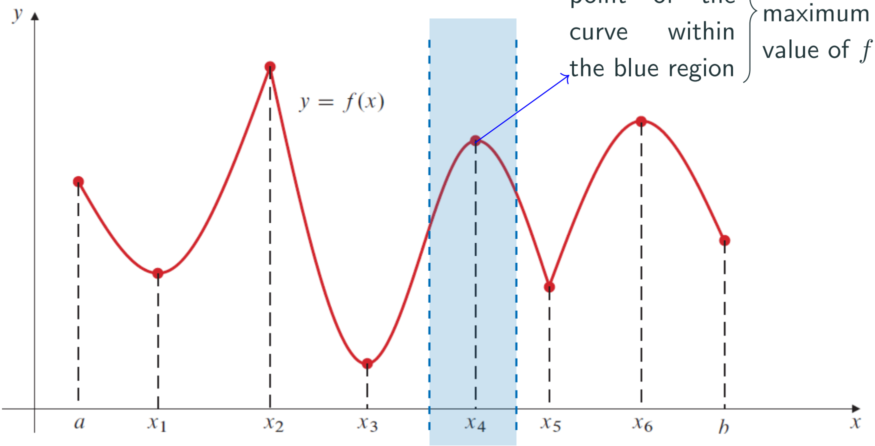
Extreme Values

the highest point of the curve within the blue region



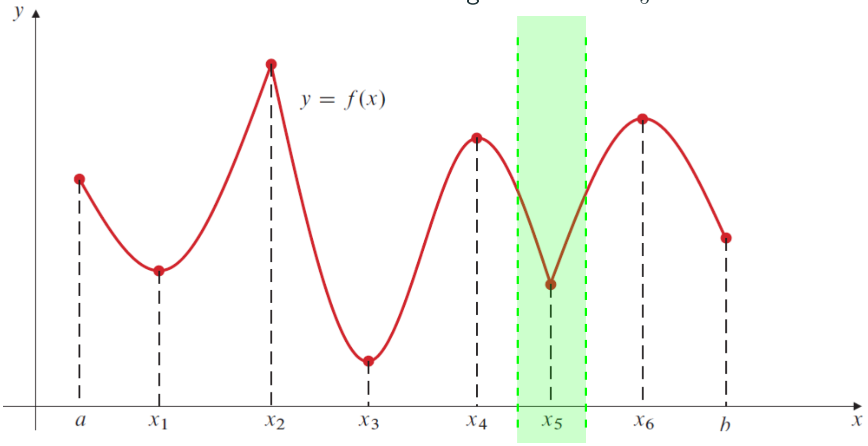
Extreme Values

the highest point of the curve within the blue region } local maximum value of f



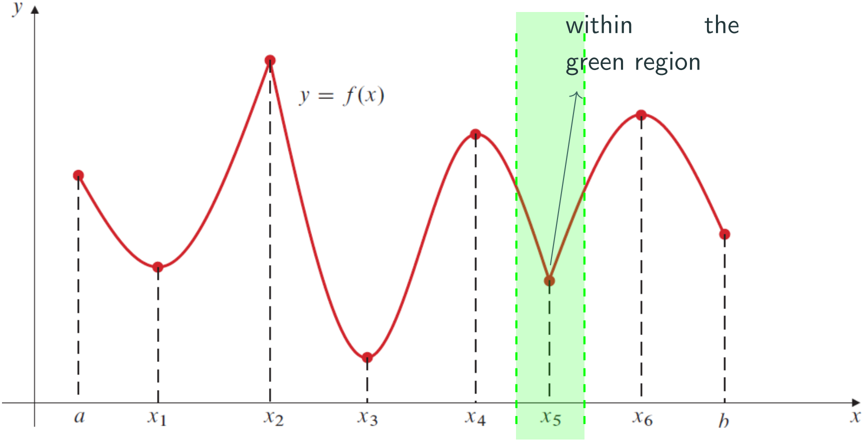
Extreme Values

Let's restrict f to a neighborhood of x_5.



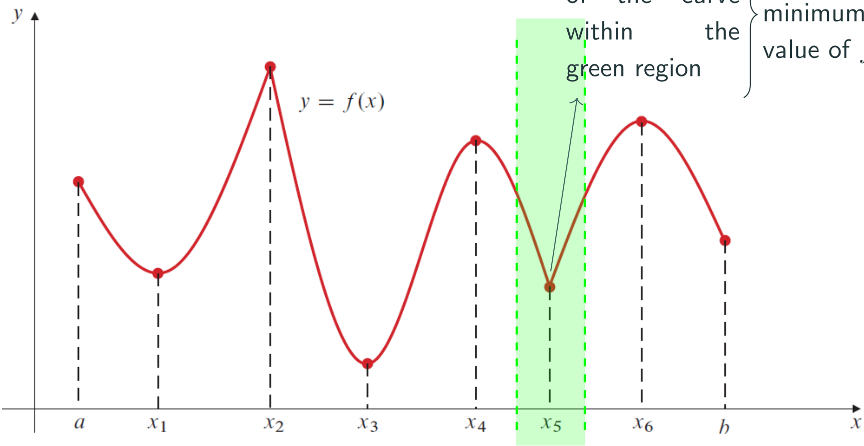
Extreme Values

the lowest point of the curve within the green region

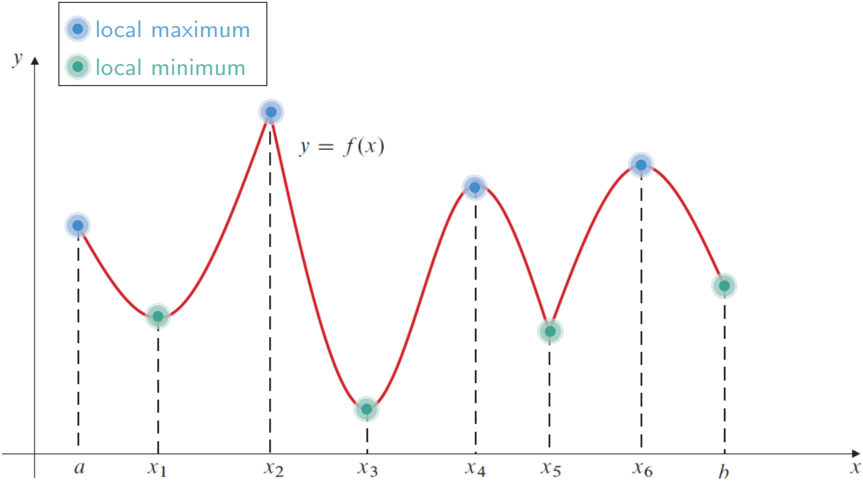


Extreme Values

the lowest point of the curve within the green region } local minimum value of f



Extreme Values



There are other local extrema of f.

Extreme Values

Definition *(Local Extreme Values)*

- Function f has a **local maximum value (loc max)** $f(x_0)$ at the point x_0 in its domain provided there exists a number $h > 0$ such that $f(x) \leq f(x_0)$ whenever x is in the domain of f and $|x - x_0| < h$.

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- Similarly, f has a **local minimum value (loc min)** $f(x_1)$ at the point x_1 in its domain provided there exists a number $h > 0$ such that $f(x) \geq f(x_1)$ whenever x is in the domain of f and $|x - x_1| < h$.