

# **MAT123 MATHEMATICS I**

## Lecture 13: Applications of Differentiation (Continued)

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## Applications of Differentiation

### Extreme Values

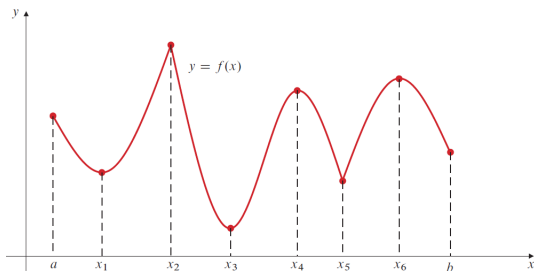
#### The First Derivative Test

### Concavity and Inflection Points

# Applications of Differentiation

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# Extreme Values



A function  $f$  can have local extreme values only at points  $x$  of three special types:

- (i) **critical points** of  $f$  (points  $x$  in  $D(f)$  where  $f'(x) = 0$ ),
- (ii) **singular points** of  $f$  (points  $x$  in  $D(f)$  where  $f'(x)$  is not defined),  
and
- (iii) **endpoints** of the domain of  $f$  (points in  $D(f)$  that do not belong to any open interval contained in  $D(f)$ ).

# Extreme Values

## Locating Extreme Values

If the function  $f$  is defined on an interval  $I$  and has a local maximum (or local minimum) value at point  $x = x_0$  in  $I$ , then  $x_0$  must be either

- (1) a critical point of  $f$ ;
- (2) a singular point of  $f$ ; or
- (3) an endpoint of  $I$ .

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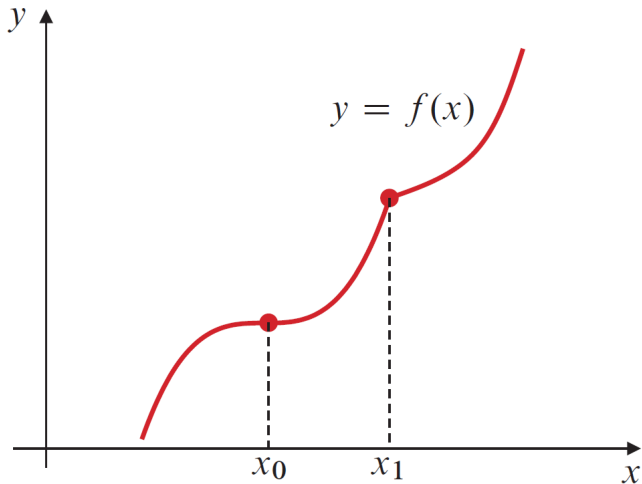
- (1) a critical point of  $f$ ;
- (2) a singular point of  $f$ ; or
- (3) an endpoint of  $I$ .

## A strategy for finding extreme values

To find the extreme values of a function  $f$  on an interval  $I$ ,

- I. Identify the critical points and endpoints of  $I$ .
- II. Evaluate  $f$  at these points.
- III. Compare the values to determine the local and absolute extrema.

# Extreme Values



A function need not have extreme values at a critical point or a singular point.

# Extreme Values

## Example

Find the maximum and minimum values of the function

$$g(x) = x^3 - 3x^2 - 9x + 2 \text{ on the interval } -2 \leq x \leq 2.$$

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## Solution

- *To find the extreme values of the function  $g(x) = x^3 - 3x^2 - 9x + 2$  on the interval  $[-2, 2]$ , we first need to find the critical points by taking the derivative and setting it to zero.*

$$g'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1) = 0$$

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- *Next, we evaluate  $g$  at the critical point and the endpoints of the interval:  $g(-2) = 0, g(-1) = 7, g(2) = -20$*

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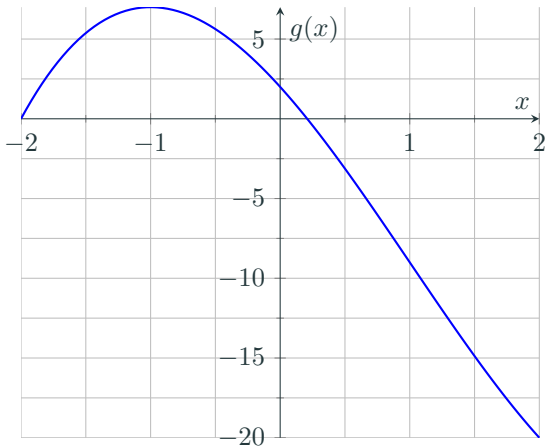
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- *Comparing these values, we find: **Maximum value:**  $g(-1) = 7$   
**Minimum value:**  $g(2) = -20$*

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• To find the extreme values of the function  $h(x) = 3x^{2/3} - 2x$  on the interval  $[-1, 1]$ , we first need to find the critical and/or singular points of  $h$  by taking the derivative.

$$h'(x) = 2x^{-1/3} - 2$$

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• Thus, we have a critical point  $x = 1$  and a singular point  $x = 0$ .

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- Thus, we have a critical point  $x = 1$  and a singular point  $x = 0$ .
- Next, we evaluate  $h$  at the singular point and the endpoints of the interval:  $h(-1) = 5$ ,  $h(0) = 0$ ,  $h(1) = 1$

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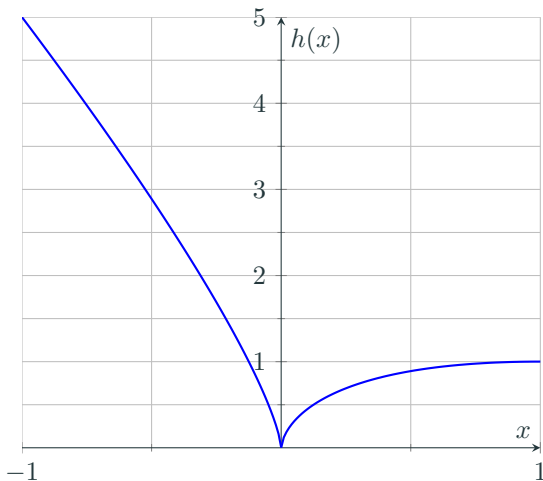
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- Comparing these values, we find: **Maximum value:**  $h(-1) = 5$   
**Minimum value:**  $h(0) = 0$

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## Example

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# Extreme Values

Theorem (*The First Derivative Test*)

## **PART I. Testing interior critical points and singular points.**

Suppose that  $f$  is continuous at  $x_0$ , and  $x_0$  is not an endpoint of the domain of  $f$ :

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Suppose  $a$  is a left endpoint of the domain of  $f$  and  $f$  is right continuous at  $a$ .

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Suppose  $b$  is a right endpoint of the domain of  $f$  and  $f$  is left continuous at  $b$ .

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● Evaluating  $f$  at the critical points and the endpoints of the interval:  $f(-2) = f(2) = 5, f(-1) = f(1) = -4, f(0) = -3$

● Comparing these values, we find:

**Absolute Maximum value:**  $f(2) = f(-2) = 5$

**Absolute Minimum value:**  $f(1) = f(-1) = -4$

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$f$  has a local maximum at  $x = \pm 2$  and  $x = 0$ , and a local minimum at  $x = \pm 1$ .

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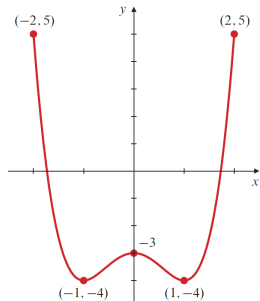
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- There is a critical point  $x = 8/27$  and a singular point  $x = 0$ .
- Evaluating  $f$  at the critical point, singular point, and the endpoints of the interval:  
 $f(-1) = -2, f(0) = 0, f(8/27) = -4/27, f(2) = 2 - 2^{2/3} \approx 0.4126$

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- Comparing these values, we find:

**Absolute Maximum value:**  $f(2) = 2 - 2^{2/3}$

**Absolute Minimum value:**  $f(-1) = -2$

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$f$  has a local maximum at  $x = 0$  and  $x = 2$ , and a local minimum at  $x = -1$  and  $x = 8/27$ .

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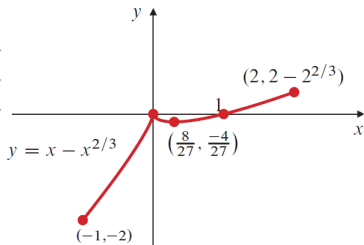
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# Extreme Values

Theorem (*Existence of extreme values on open intervals*)

If  $f$  is continuous on the open interval  $(a, b)$ , and if

$$\lim_{x \rightarrow a^+} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow b^-} f(x) = M,$$

then the following conclusions hold:

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then the following conclusions hold:

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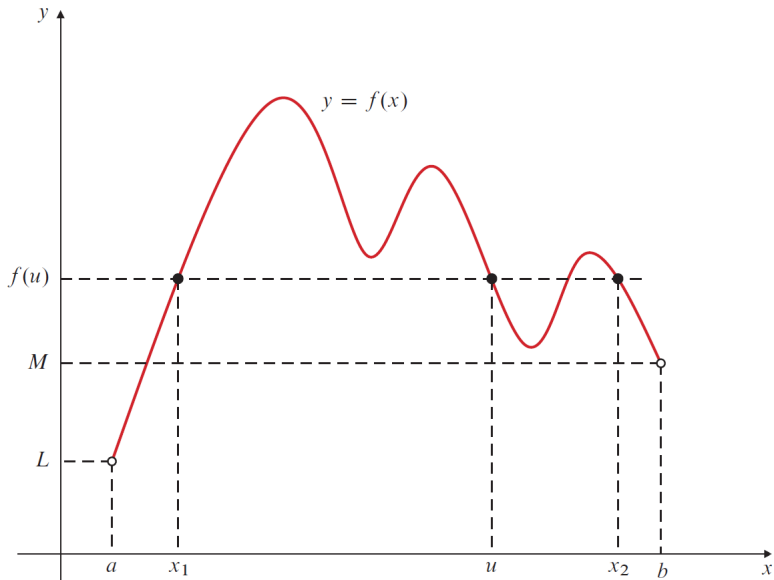
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In the above theorem,  $a$  may be equal to  $-\infty$ , where  $\lim_{x \rightarrow a^+}$  should be replaced by  $\lim_{x \rightarrow -\infty}$  and  $b$  may be equal to  $+\infty$ , where  $\lim_{x \rightarrow b^-}$  should be replaced by  $\lim_{x \rightarrow +\infty}$ . Also, either or both of  $L$  and  $M$  may be either  $-\infty$  or  $\infty$ .

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## Example

Show that  $f(x) = x + \frac{4}{x}$  has an absolute minimum value on the interval  $(0, \infty)$ , and find that minimum value.

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## Solution

● To find the extreme values of the function  $f(x) = x + \frac{4}{x}$  on the interval  $(0, \infty)$ , we first need to find the critical points by taking the

derivative.  $f'(x) = 1 - \frac{4}{x^2} = \frac{(x-2)(x+2)}{x^2}$

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$$f(2) = 4, \lim_{x \rightarrow 0^+} f(x) = +\infty, \lim_{x \rightarrow +\infty} f(x) = +\infty$$

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- Since  $f(2) = 4$  is less than both limits, we conclude that:  $f$  has an absolute minimum value of 4 at  $x = 2$ .

# Extreme Values

## Example

Let  $f(x) = xe^{-x^2}$ . Find and classify the critical points of  $f$ , evaluate  $\lim_{x \rightarrow 0^+} f(x)$  and  $\lim_{x \rightarrow \pm\infty} f(x)$ .

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- Since  $f(1/\sqrt{2}) = \frac{1}{\sqrt{2}}e^{-1/2} = \frac{1}{\sqrt{2}e} > 0$ ,  $f$  has an **absolute maximum**.

# Extreme Values

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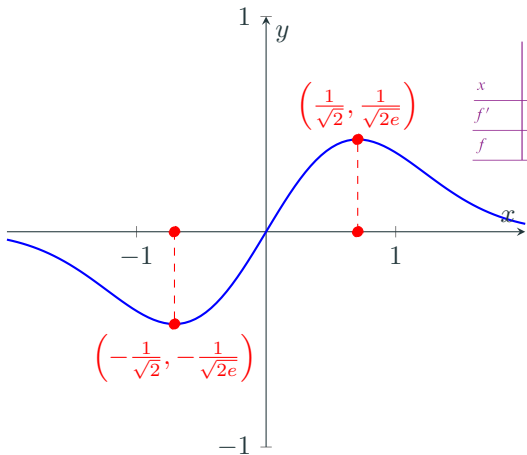
- Since  $f(1/\sqrt{2}) = \frac{1}{\sqrt{2}}e^{-1/2} = \frac{1}{\sqrt{2e}} > 0$ ,  $f$  has an **absolute maximum**.
- Since  $f(-1/\sqrt{2}) = -\frac{1}{\sqrt{2}}e^{-1/2} = -\frac{1}{\sqrt{2e}} < 0$ ,  $f$  has an **absolute minimum**.

# Extreme Values

## Example

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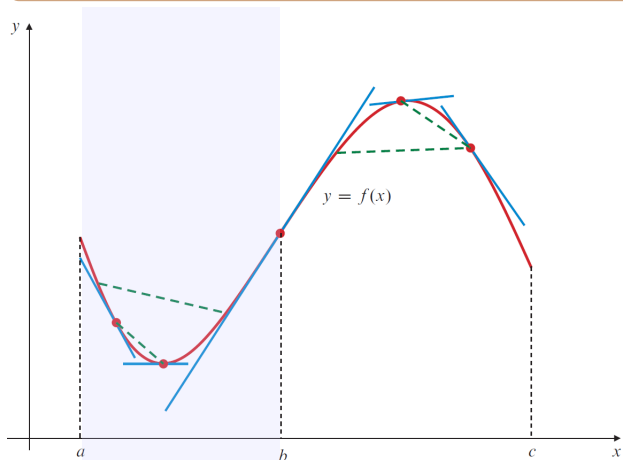


$x$		CP		CP	
		$-1/\sqrt{2}$		$1/\sqrt{2}$	
$f'$	-	0	+	0	-
$f$	$\searrow$	min	$\nearrow$	max	$\searrow$

# Concavity and Inflection Points

## Concavity of the function $f$

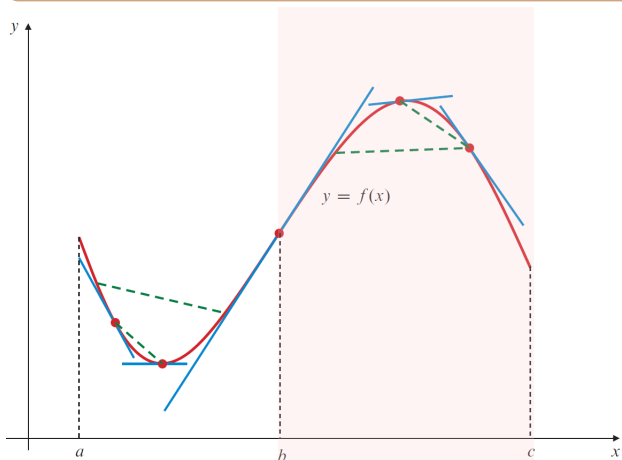
We say that  $f$  is **concave up** on an open interval  $I$  if it is differentiable there and the derivative  $f'$  is an increasing function on  $I$ .



# Concavity and Inflection Points

Concavity of the function  $f$

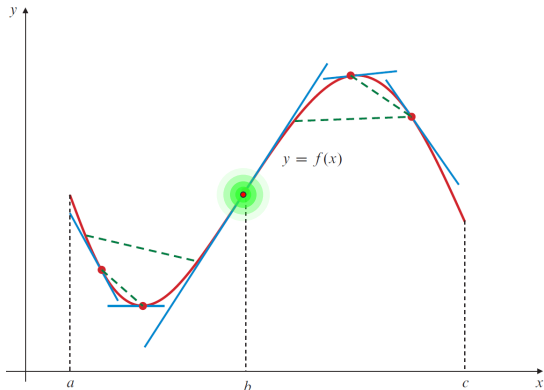
Similarly,  $f$  is **concave down** on  $I$  if  $f'$  exists and is decreasing on  $I$ .



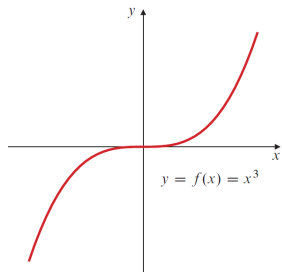
# Concavity and Inflection Points

## Concavity of the function $f$

We say that  $f$  has an **inflection point** at  $x_0$  if the following two conditions are satisfied: **(a)**  $y = f(x)$  has a tangent line at  $x = x_0$ , and **(b)** the concavity of  $f$  is opposite on opposite sides of  $x_0$ .

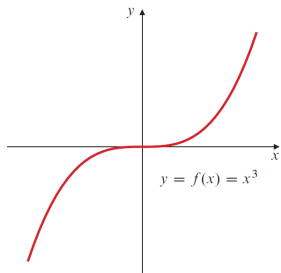


# Concavity and Inflection Points

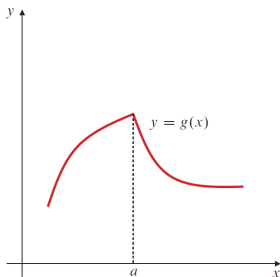


$x = 0$  is a critical point  
of  $f$ , and  $f$  has an  
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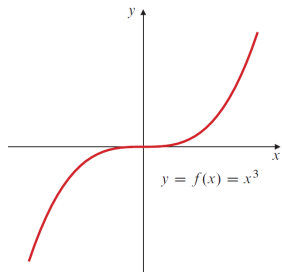


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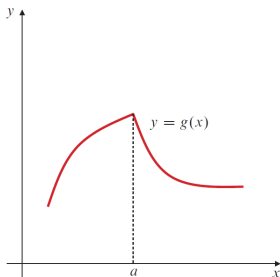


The concavity of  $g$  is opposite on opposite sides of the singular point  $a$ , but its graph has no tangent and therefore no inflection point there.

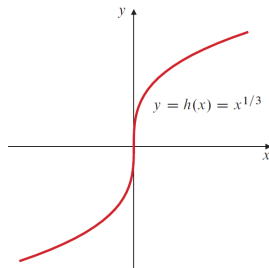
# Concavity and Inflection Points



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This graph of  $h$  has an inflection point at the origin even though  $x = 0$  is a singular point of  $h$ .

# Concavity and Inflection Points

## Concavity and the second derivative

- (a) If  $f''(x) > 0$  on interval  $I$ , then  $f$  is concave up on  $I$ .
- (b) If  $f''(x) < 0$  on interval  $I$ , then  $f$  is concave down on  $I$ .
- (c) If  $f$  has an inflection point at  $x_0$  and  $f''(x_0)$  exists, then  $f''(x_0) = 0$ .

# Concavity and Inflection Points

## Example

Determine the intervals of increase and decrease, the local extreme values, the intervals of concavity, and the inflection points of the function  $f(x) = x^4 - 2x^3 + 1$ . Use the information to sketch the graph of  $f$ .

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### Solution

$$f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3) = 0 \text{ at } x = 0 \text{ and } x = \frac{3}{2}.$$

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Analyzing the signs of  $f'$  and  $f''$ :

$x$		CP			CP		
		0		1		3/2	
$f'$	-	0	-		-	0	+
$f''$	+	0	-	0	+		+
$f$	$\searrow$		$\searrow$		$\searrow$	min	$\nearrow$
	$\smile$	infl	$\frown$	infl	$\smile$		$\smile$

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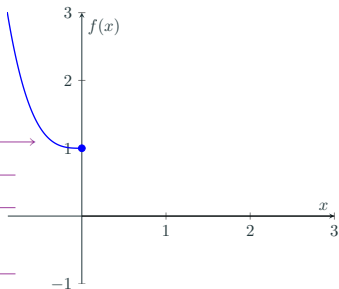
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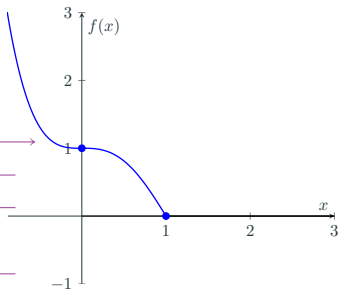
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$f$	$\searrow$		$\searrow$	$\searrow$	min	$\nearrow$
	$\cup$	infl	$\cap$	infl	$\cup$	$\cup$



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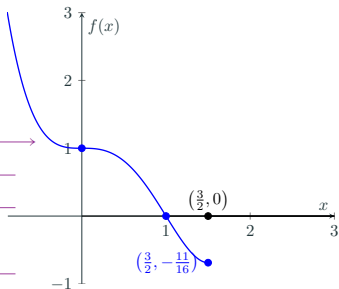
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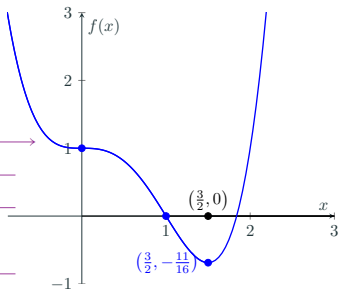
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	$\cup$	infl	$\cap$	infl	$\cup$		$\cap$



# Concavity and Inflection Points

## Theorem (*The Second Derivative Test*)

**(a)** If  $f'(x_0) = 0$  and  $f''(x_0) < 0$ , then  $f$  has a local maximum value at  $x_0$ .

**(b)** If  $f'(x_0) = 0$  and  $f''(x_0) > 0$ , then  $f$  has a local minimum value at  $x_0$ .

**(c)** If  $f'(x_0) = 0$  and  $f''(x_0) = 0$ , then the test is inconclusive;  $f$  may have a local maximum at  $x_0$  or a local minimum, or it may have an inflection point instead.

# Concavity and Inflection Points

## Example

Find and classify the critical points of  $f(x) = x^2e^{-x}$ .

# Concavity and Inflection Points

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Find and classify the critical points of  $f(x) = x^2e^{-x}$ .

## Solution

*We begin by computing the first two derivatives:*

$$f'(x) = 2xe^{-x} - x^2e^{-x} = e^{-x}(2x - x^2) = 0 \text{ at } x = 0 \text{ and } x = 2.$$

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*Evaluating  $f''(0)$  and  $f''(2)$ :*

$$f''(0) = 2 > 0 \implies \text{local minimum at } x = 0.$$

*and*

$$f''(2) = e^{-2}(-2) < 0 \implies \text{local maximum at } x = 2.$$