MAT123 MATHEMATICS I

Lecture 14: Applications of Differentiation (Continued)

Outline

Applications of Differentiation

Sketching the Graph of a Function: Asymptotes

- **Vertical Asymptotes:** Occur when the function approaches infinity as x approaches a certain value.
- **Horizontal Asymptotes:** Occur when the function approaches a constant value as x approaches infinity or negative infinity.
- **Oblique Asymptotes:** Occur when the function approaches a linear function as x approaches infinity or negative infinity.

Sketching the Graph of a Function: Asymptotes

Vertical Asymptotes

The graph of $y = f(x)$ has a **vertical asymptote** at $x = a$ if either

$$\lim_{x \to a^-} f(x) = \pm\infty$$

or

$$\lim_{x \to a^+} f(x) = \pm\infty,$$

or both.

Sketching the Graph of a Function: Asymptotes

Example

Find the vertical asymptotes of $f(x) = \dfrac{1}{x^2 - x}$. How does the graph approach these asymptotes?

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To see how the graph approaches these asymptotes, we can analyze the limits:

$$\lim_{x \to 0^-} f(x) = \infty, \quad \lim_{x \to 0^+} f(x) = -\infty$$

$$\lim_{x \to 1^-} f(x) = -\infty, \quad \lim_{x \to 1^+} f(x) = +\infty$$

Sketching the Graph of a Function: Asymptotes

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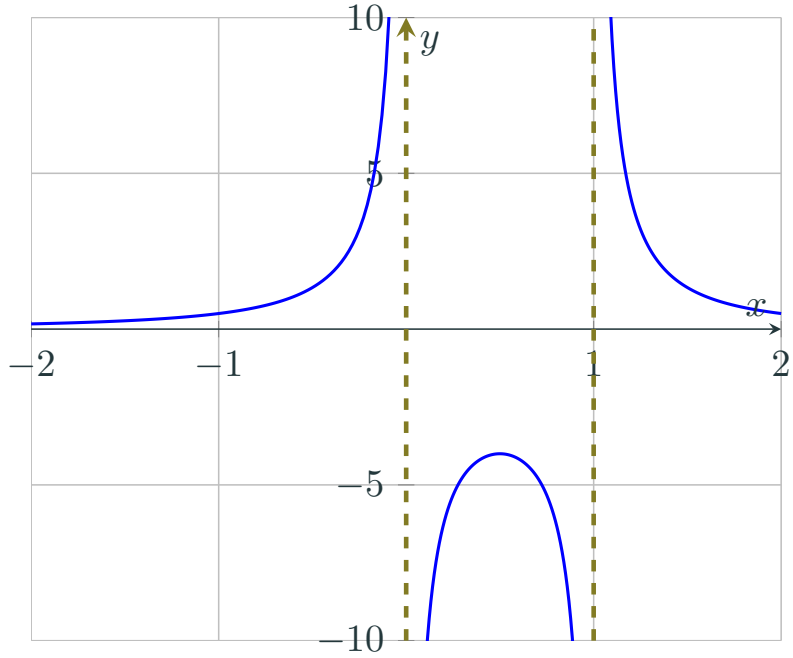


Sketching the Graph of a Function: Asymptotes

Horizontal Asymptotes

The graph of $y = f(x)$ has a **horizontal asymptote** at $y = b$ if either

$$\lim_{x \to \infty} f(x) = b$$

or

$$\lim_{x \to -\infty} f(x) = b,$$

or both.

Sketching the Graph of a Function: Asymptotes

Example

Find the horizontal asymptotes of

(a) $f(x) = \dfrac{1}{x^2 - x}$ **(b)** $f(x) = \dfrac{x^4 + x^2}{x^4 + 1}$.

Sketching the Graph of a Function: Asymptotes

Example

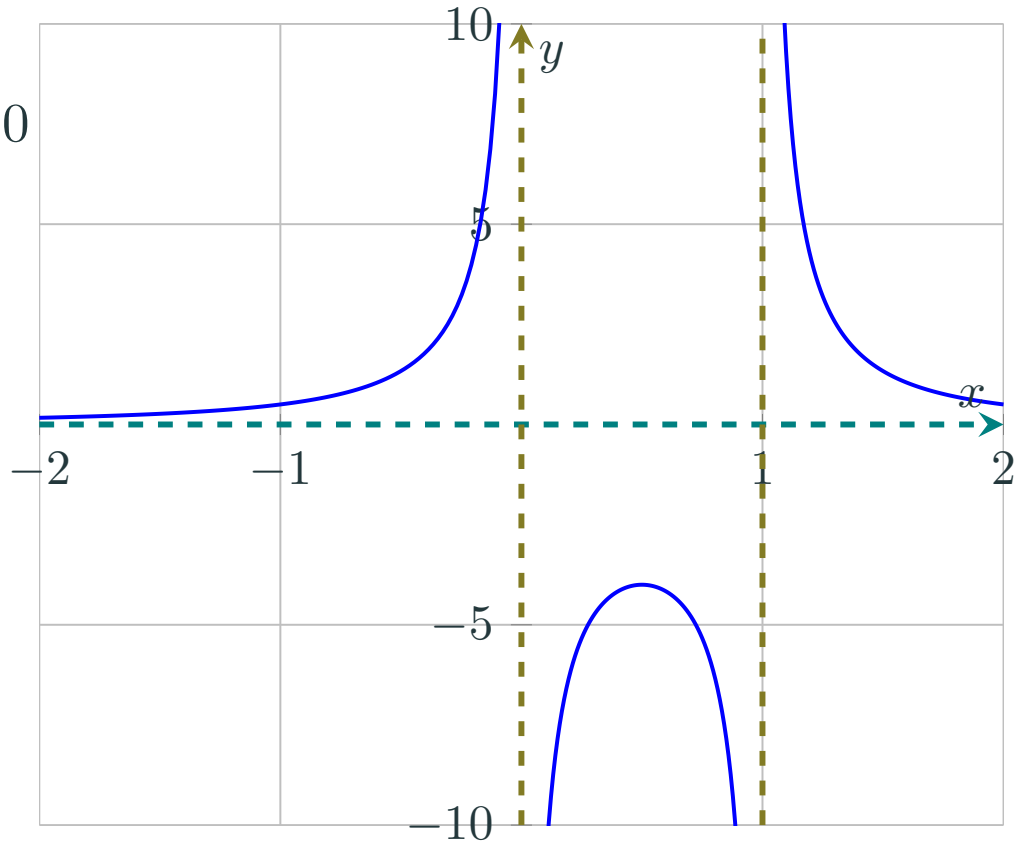
Find the horizontal asymptotes of

(a) $f(x) = \dfrac{1}{x^2 - x}$ **(b)** $f(x) = \dfrac{x^4 + x^2}{x^4 + 1}$.

Solution

(a) $f(x) = \frac{1}{x^2 - x}$ *has a horizontal asymptote at* $y = 0$ *since*

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \infty} \frac{1}{x^2 - x} = 0$$



Sketching the Graph of a Function: Asymptotes

Example

Find the horizontal asymptotes of

(a) $f(x) = \dfrac{1}{x^2 - x}$ **(b)** $f(x) = \dfrac{x^4 + x^2}{x^4 + 1}$.

Solution

(b) $f(x) = \frac{x^4+x^2}{x^4+1}$ *has a horizontal asymptote at* $y = 1$ *since*

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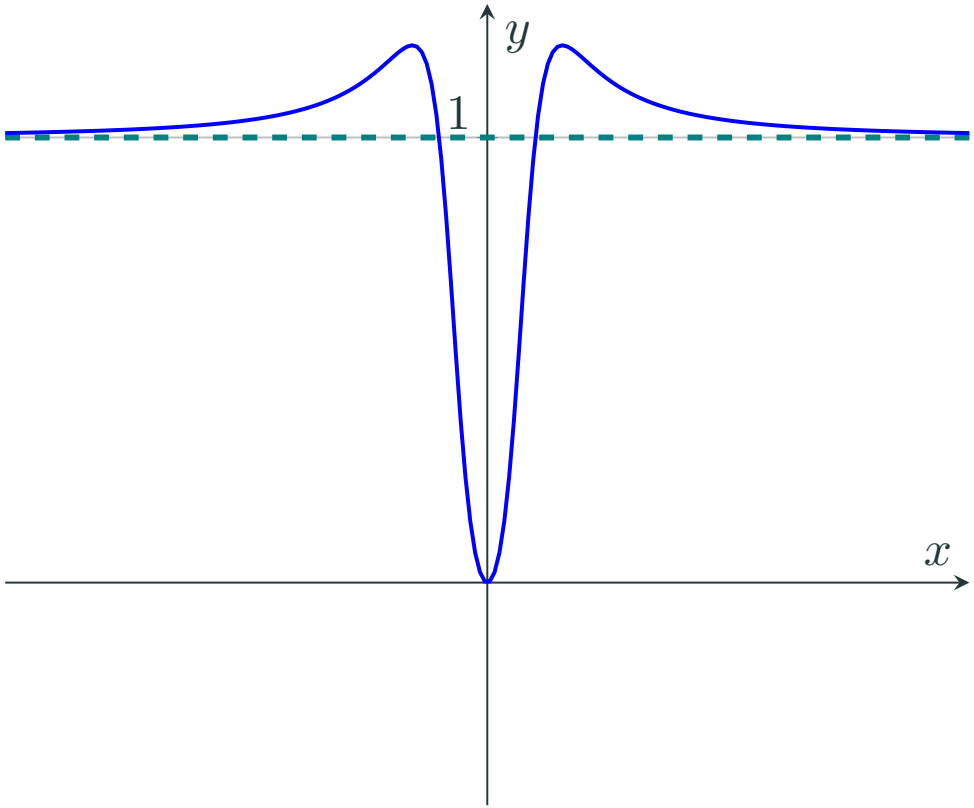

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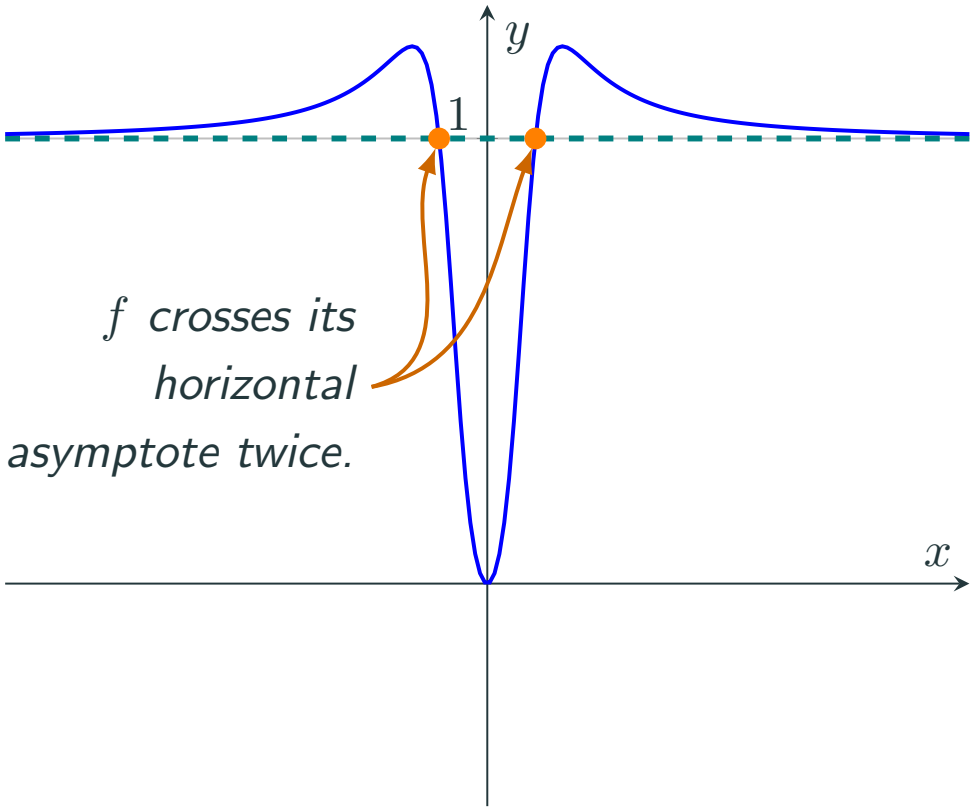

Sketching the Graph of a Function: Asymptotes

Oblique Asymptotes

The graph of $y = f(x)$ has an **oblique asymptote** at $y = mx + b$ if either

$$\lim_{x \to \infty} (f(x) - (mx + b)) = 0$$

or

$$\lim_{x \to -\infty} (f(x) - (mx + b)) = 0,$$

or both.

Sketching the Graph of a Function: Asymptotes

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Remark

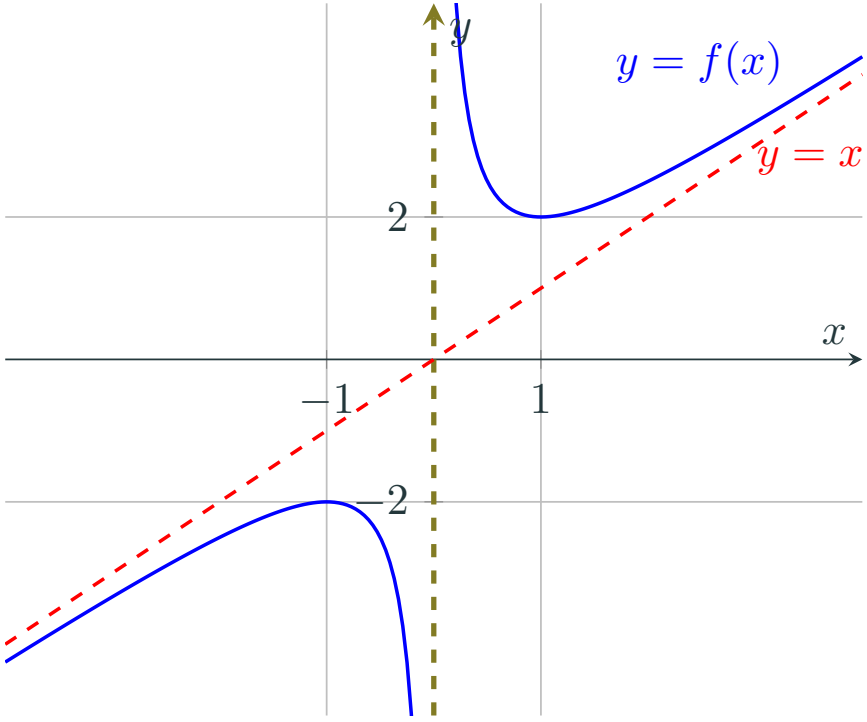
Note that for an oblique asymptote, the slope m cannot be zero, as that would correspond to a horizontal asymptote. The coefficients in $y = mx + b$ can be determined as follows:

$$m = \lim_{x \to \pm\infty} \frac{f(x)}{x} \text{ and } b = \lim_{x \to \pm\infty} (f(x) - mx).$$

Sketching the Graph of a Function: Asymptotes

Example

Consider the function $f(x) = \dfrac{x^2+1}{x} = x + \dfrac{1}{x}$, whose graph is shown below.



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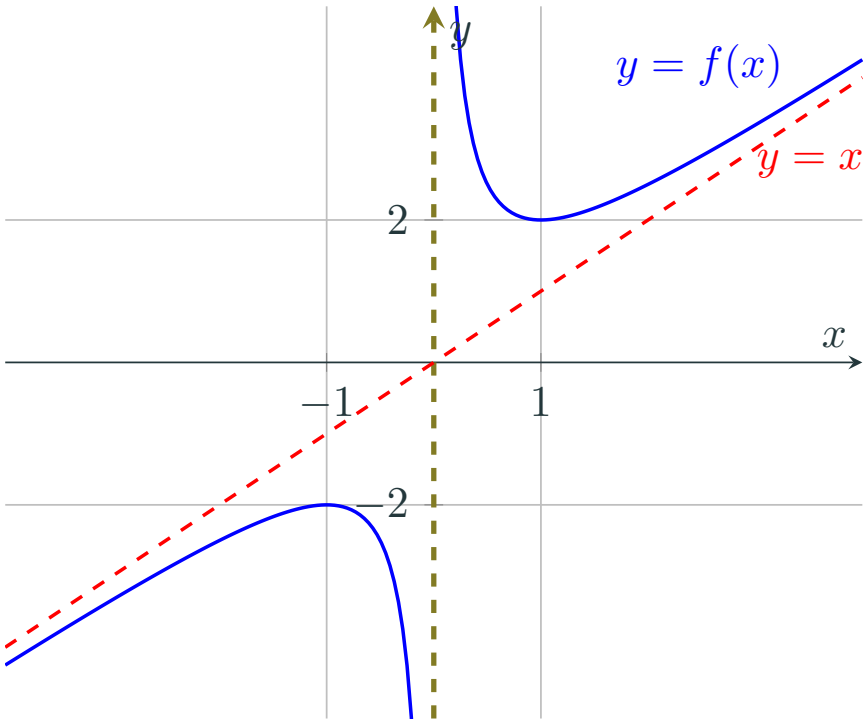

$$\lim_{x\to\pm\infty} (f(x) - x) = \lim_{x\to\pm\infty} \frac{1}{x} = 0.$$

Sketching the Graph of a Function: Asymptotes

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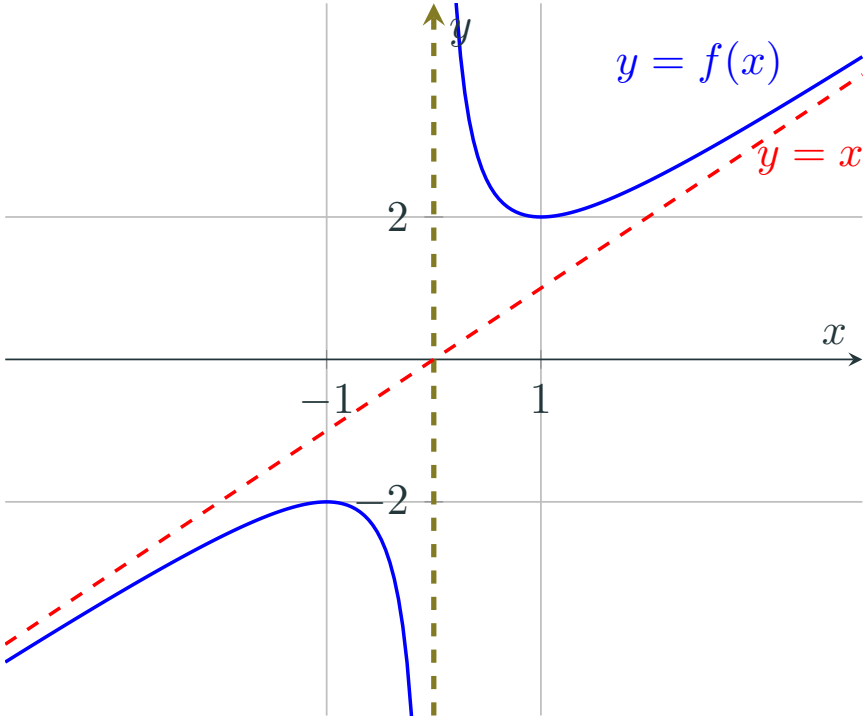

$$\lim_{x\to\pm\infty} (f(x) - x) = \lim_{x\to\pm\infty} \frac{1}{x} = 0.$$

Therefore, the straight line $y = x$ is a two-sided oblique asymptote of f.

Sketching the Graph of a Function: Asymptotes

Example

The graph of $y = \dfrac{xe^x}{1+e^x}$ has a horizontal asymptote at $y = 0$ at the left and since

$$\lim_{x\to-\infty} \frac{xe^x}{1+e^x} = \frac{0}{1} = 0$$

and an oblique asymptote $y = x$ at the right since

$$\lim_{x\to\infty} \left(\frac{xe^x}{1+e^x} - x \right) = \lim_{x\to\infty} \frac{xe^x - x(1+e^x)}{1+e^x}$$

$$= \lim_{x\to\infty} \frac{-x}{1+e^x} = 0.$$

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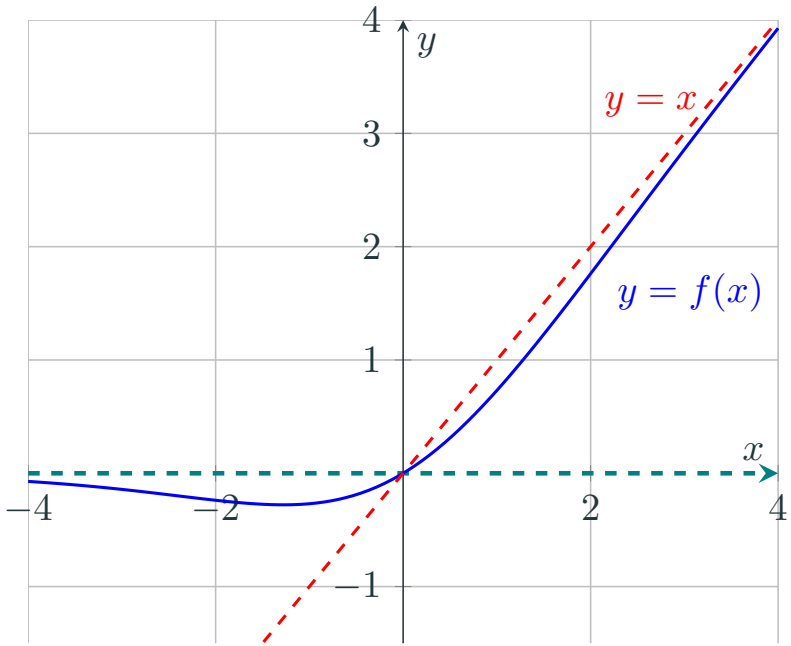

Sketching the Graph of a Function: Asymptotes

Suppose that $f(x) = \dfrac{P_m(x)}{Q_n(x)}$, where P_m and Q_n are polynomials of degree m and n, respectively. Suppose also that P_m and Q_n have no common factors. Then:

Sketching the Graph of a Function: Asymptotes

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(b) The graph of f has a two-sided horizontal asymptote $y = 0$ if $m < n$.

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(c) The graph of f has a horizontal asymptote $y = L,\ (L \neq 0)$ if $m = n$. L is the quotient of the leading coefficients of P_m and Q_n.

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(d) The graph of f has a two-sided oblique asymptote if $m = n + 1$. This asymptote can be found by performing polynomial long division to obtain a linear quotient $y = mx + b$, and remainder $R(x)$ such that

$$f(x) = mx + b + \frac{R(x)}{Q_n(x)}.$$

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$$f(x) = mx + b + \frac{R(x)}{Q_n(x)}.$$

(e) The graph of f has no horizontal or oblique asymptotes if $m > n + 1$.

Sketching the Graph of a Function: Asymptotes

Example

Find the oblique asymptote of $y = \dfrac{x^3}{x^2 + x + 1}$.

Solution

To find the oblique asymptote, we perform polynomial long division on x^3 *by* $x^2 + x + 1$.

$$\begin{array}{rl|l} x^3 & & x^2 + x + 1 \\ \cline{3-3} -x^3 - x^2 - x & & x - 1 \\ \cline{1-1} \quad -x^2 - x & & \\ \quad x^2 + x + 1 & & \\ \cline{1-1} \quad 1 & & \end{array}$$

The quotient is $y = x - 1$ *and the remainder is* $R(x) = 1$. *Thus, the oblique asymptote is* $y = x - 1$.

Sketching the Graph of a Function: Examples

Checklist for curve sketching

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1. Calculate $f'(x)$ and $f''(x)$, and express the results in factored form.

Sketching the Graph of a Function: Examples

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   2.1 Any vertical asymptotes. (Look for zeros of denominators.)

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   - 2.4 Any easily calculated intercepts (points with coordinates $(x, 0)$ or $(0, y)$) or endpoints or other "obvious" points.

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3. Examine $f'(x)$ for the following:
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3. Examine $f'(x)$ for the following:
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   - 3.2 Any points where f' is not defined. (These will include singular points, endpoints of the domain of f; and vertical asymptotes.)
   - 3.3 Intervals on which f' is positive or negative.

Sketching the Graph of a Function: Examples

Checklist for curve sketching

4. Examine $f''(x)$ for the following:
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Sketching the Graph of a Function: Examples

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Sketching the Graph of a Function: Examples

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   - 4.4 Any inflection points.

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 + 2x + 4}{2x}$

Solution

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 + 2x + 4}{2x}$

Solution

Step 1: *Find the domain of f.*

$$f(x) = \frac{x^2 + 2x + 4}{2x} \quad \textit{is defined for } x \neq 0.$$

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 + 2x + 4}{2x}$

Solution

Step 1: *Domain of f is* $(-\infty, 0) \cup (0, \infty)$

Step 2: *Find the vertical asymptote. The function has a vertical asymptote at* $x = 0$ *since*

$$\lim_{x \to 0^-} f(x) = -\infty \quad \textit{and} \quad \lim_{x \to 0^+} f(x) = +\infty.$$

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Step 1: *Domain of f is* $(-\infty, 0) \cup (0, \infty)$

Step 2: *The function has a vertical asymptote at* $x = 0$

Step 3: *Find the horizontal asymptotes.*

$$\lim_{x \to -\infty} f(x) = -\infty \text{ and } \lim_{x \to \infty} f(x) = \infty.$$

Thus, there is no horizontal asymptote.

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 + 2x + 4}{2x}$

Solution

Step 1: *Domain of f is $(-\infty, 0) \cup (0, \infty)$*

Step 2: *The function has a vertical asymptote at $x = 0$*

Step 3: *No horizontal asymptotes.*

Step 4: *Find oblique asymptotes. Since $y = \frac{x}{2} + 1 + \frac{2}{x}$, there is an oblique asymptote $y = \frac{x}{2} + 1$.*

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Sketch the graph of $y = \dfrac{x^2 + 2x + 4}{2x}$

Solution

Step 1: *Domain of f is* $(-\infty, 0) \cup (0, \infty)$

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Step 3: *No horizontal asymptotes.*

Step 4: *There is an oblique asymptote* $y = \frac{x}{2} + 1$.

Step 5: *Examine* $f'(x)$.

$$f'(x) = \frac{x^2 - 4}{2x^2} = 0 \Rightarrow x = \pm 2$$

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Step 6: *Examine* $f''(x)$.

$$f''(x) = \frac{4}{x^3}$$

is undefined when $x = 0$.

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Sketch the graph of $y = \dfrac{x^2 + 2x + 4}{2x}$

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		CP		ASY		CP	
x		-2		0		2	
y'	$+$	0	$-$	undef	$-$	0	$+$
y''	$-$		$-$	undef	$+$		$+$
y	$\nearrow$	max	$\searrow$	undef	$\searrow$	min	$\nearrow$
	$\frown$		$\frown$		$\smile$		$\smile$

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$$f''(x) = \frac{4}{x^3}$$

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Example

Sketch the graph of $y = \dfrac{x^2 + 2x + 4}{2x}$

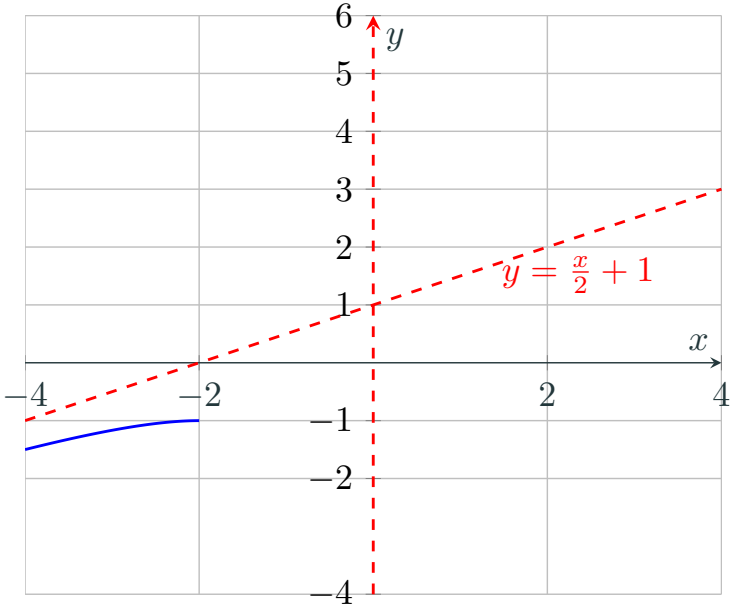
Solution

Step 1: *Domain of f is* $(-\infty, 0) \cup (0, \infty)$

Step 2: *The function has a vertical asymptote at* $x = 0$

Step 3: *No horizontal asymptotes.*

Step 4: *There is an oblique asymptote* $y = \frac{x}{2} + 1$.



		CP		ASY		CP	
x		-2		0		2	
y'	+	0	−	undef	−	0	+
y''	−		−	undef	+		+
y	↗	max	↘	undef	↘	min	↗
	⌒		⌒		⌣		⌣

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Sketch the graph of $y = \dfrac{x^2 + 2x + 4}{2x}$

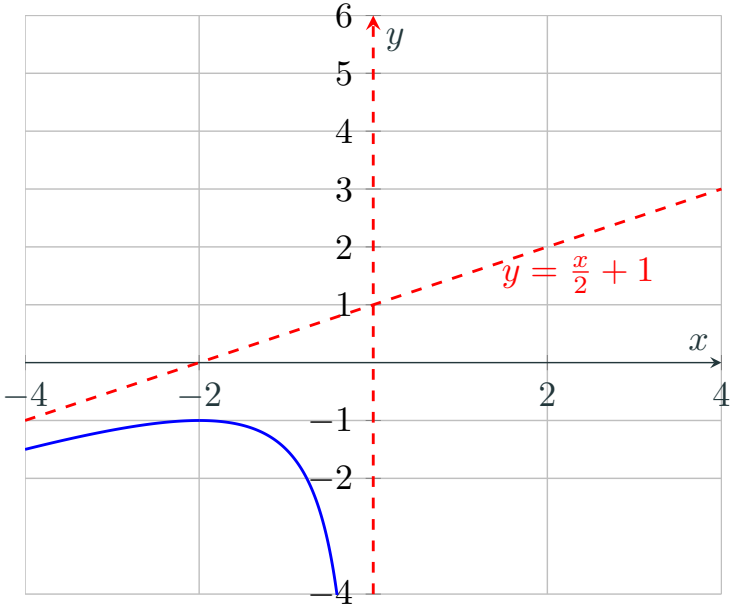
Solution

Step 1: *Domain of f is* $(-\infty, 0) \cup (0, \infty)$

Step 2: *The function has a vertical asymptote at* $x = 0$

Step 3: *No horizontal asymptotes.*

Step 4: *There is an oblique asymptote* $y = \frac{x}{2} + 1$.



		CP		ASY		CP	
x		-2		0		2	
y'	$+$	0	$-$	undef	$-$	0	$+$
y''	$-$		$-$	undef	$+$		$+$
y	$\nearrow$	max	$\searrow$	undef	$\searrow$	min	$\nearrow$
	$\frown$		$\frown$		$\smile$		$\smile$

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 + 2x + 4}{2x}$

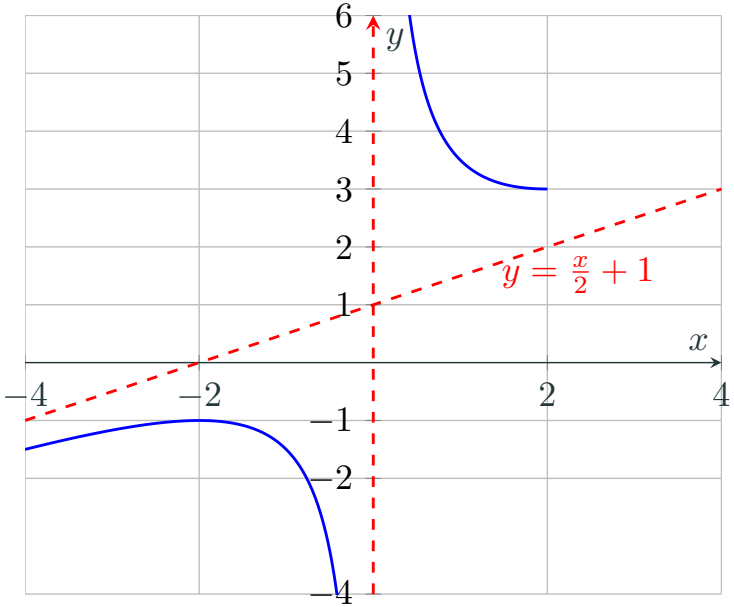
Solution

Step 1: *Domain of f is* $(-\infty, 0) \cup (0, \infty)$

Step 2: *The function has a vertical asymptote at* $x = 0$

Step 3: *No horizontal asymptotes.*

Step 4: *There is an oblique asymptote* $y = \frac{x}{2} + 1$.



		CP		ASY		CP	
x		-2		0		2	
y'	$+$	0	$-$	undef	$-$	0	$+$
y''	$-$		$-$	undef	$+$		$+$
y	$\nearrow$	max	$\searrow$	undef	$\searrow$	min	$\nearrow$
	$\frown$		$\frown$		$\smile$		$\smile$

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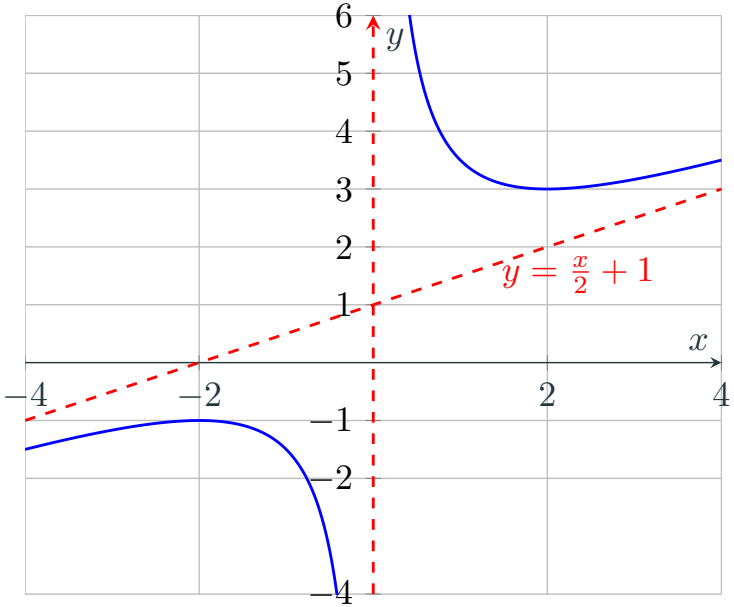

		CP		ASY		CP	
x		-2		0		2	
y'	$+$	0	$-$	undef	$-$	0	$+$
y''	$-$		$-$	undef	$+$		$+$
y	$\nearrow$	max	$\searrow$	undef	$\searrow$	min	$\nearrow$
	$\frown$		$\frown$		$\smile$		$\smile$

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$

Solution

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$

Solution

Step 1: *Find the domain of f.*

$$f(x) = \frac{x^2 - 1}{x^2 - 4} = \frac{(x-1)(x+1)}{(x-2)(x+2)} \quad \textit{is defined for } x \neq \pm 2.$$

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$

Solution

Step 1: *Domain of f is* $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Step 2: *Find the vertical asymptotes. The function has vertical asymptotes at* $x = \pm 2$ *since*

$$\lim_{x \to 2^-} f(x) = -\infty, \quad \lim_{x \to 2^+} f(x) = \infty$$

$$\lim_{x \to (-2)^-} f(x) = \infty, \quad \lim_{x \to (-2)^+} f(x) = -\infty$$

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$

Solution

Step 1: *Domain of f is* $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Step 2: *The function has vertical asymptotes at* $x = \pm 2$

Step 3: *Find the horizontal asymptotes.*

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^2 - 1}{x^2 - 4} = 1.$$

Thus, there is a horizontal asymptote at $y = 1$.

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$

Solution

Step 1: *Domain of f is* $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Step 2: *The function has vertical asymptotes at* $x = \pm 2$

Step 3: *Horizontal asymptote:* $y = 1$.

Step 4: *Find intercepts. y-intercept:* $f(0) = \frac{1}{4}$. *x-intercepts:* $x = \pm 1$.

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$

Solution

Step 1: *Domain of f is* $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Step 2: *The function has vertical asymptotes at* $x = \pm 2$

Step 3: *Horizontal asymptote:* $y = 1$.

Step 4: *Intercepts:* $(0, \frac{1}{4})$, $(\pm 1, 0)$.

Step 5: *Examine* $f'(x)$.

$$f'(x) = \frac{-6x}{(x^2 - 4)^2} = 0 \Rightarrow x = 0$$

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$

Solution

Step 1: *Domain of f is* $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Step 2: *The function has vertical asymptotes at* $x = \pm 2$

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Step 5: *Examine* $f'(x)$.

$$f'(x) = \frac{-6x}{(x^2 - 4)^2} = 0 \Rightarrow x = 0$$

Step 6: *Examine* $f''(x)$.

$$f''(x) = \frac{6(3x^2 + 4)}{(x^2 - 4)^3}$$

changes sign at $x = \pm 2$.

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$

Solution

Step 1: *Domain of f is* $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Step 2: *The function has vertical asymptotes at* $x = \pm 2$

Step 3: *Horizontal asymptote:* $y = 1$.

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		ASY		CP		ASY	
x		-2		0		2	
f'	+	undef	+	0	−	undef	−
f''	+	undef	−		−	undef	+
f	↗	undef	↗	max	↘	undef	↘
	⌣		⌢		⌢		⌣

Step 6: *Examine* $f''(x)$.

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Sketching the Graph of a Function: Examples

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Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$

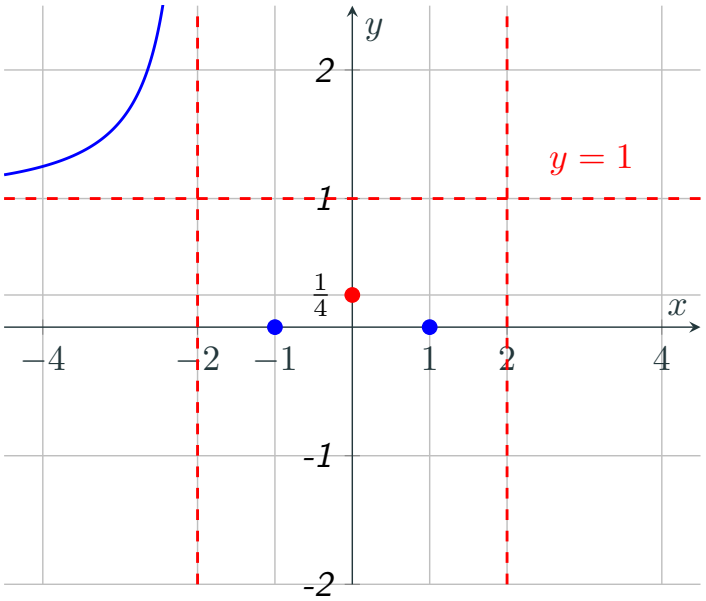
Solution

Step 1: *Domain of f is* $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Step 2: *The function has vertical asymptotes at* $x = \pm 2$

Step 3: *Horizontal asymptote:* $y = 1$.

Step 4: *Intercepts:* $(0, \frac{1}{4})$, $(\pm 1, 0)$.



		ASY		CP		ASY	
x		-2		0		2	
f'	$+$	undef	$+$	0	$-$	undef	$-$
f''	$+$	undef	$-$		$-$	undef	$+$
f	↗	undef	↗	max	↘	undef	↘
	⌒		⌒		⌒		⌒

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$

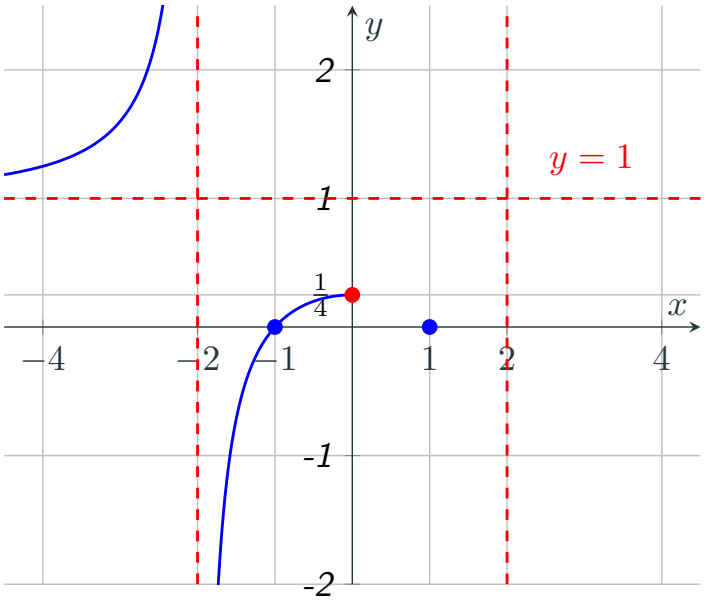
Solution

Step 1: *Domain of f is* $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Step 2: *The function has vertical asymptotes at* $x = \pm 2$

Step 3: *Horizontal asymptote:* $y = 1$.

Step 4: *Intercepts:* $(0, \frac{1}{4})$, $(\pm 1, 0)$.



		ASY		CP		ASY	
x		-2		0		2	
f'	+	undef	+	0	−	undef	−
f''	+	undef	−		−	undef	+
f	↗	undef	↗	max	↘	undef	↘
	⌣		⌢		⌢		⌣

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$

Solution

Step 1: *Domain of f is* $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

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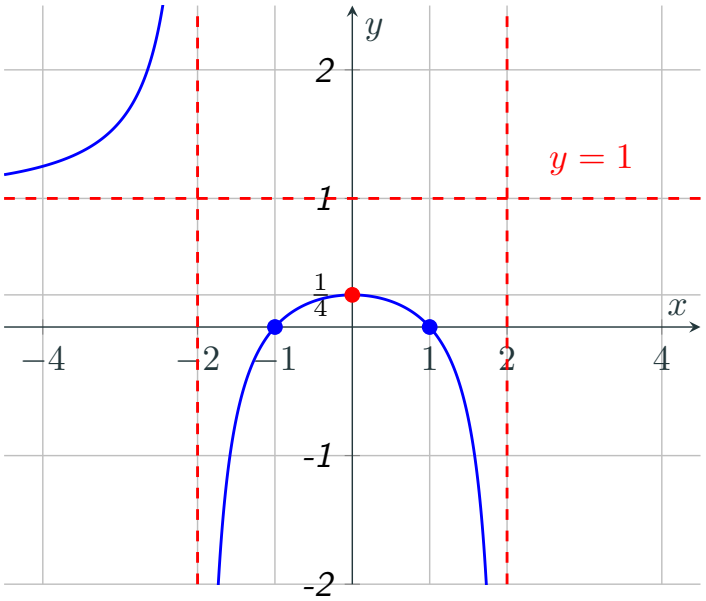
Step 3: *Horizontal asymptote:* $y = 1$.

Step 4: *Intercepts:* $(0, \frac{1}{4})$, $(\pm 1, 0)$.



		ASY		CP		ASY	
x		-2		0		2	
f'	$+$	undef	$+$	0	$-$	undef	$-$
f''	$+$	undef	$-$		$-$	undef	$+$
f	$\nearrow$	undef	$\nearrow$	max	$\searrow$	undef	$\searrow$
	$\smile$		$\frown$		$\frown$		$\smile$

Sketching the Graph of a Function: Examples

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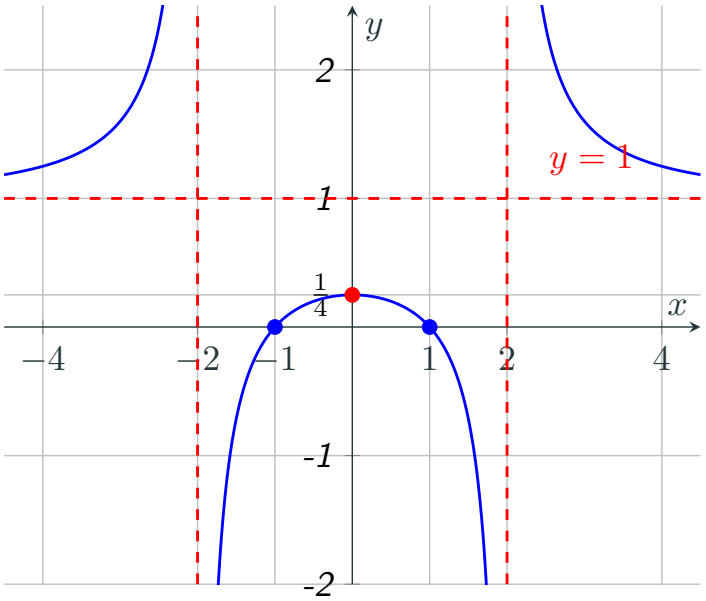

		ASY		CP		ASY	
x		-2		0		2	
f'	$+$	undef	$+$	0	$-$	undef	$-$
f''	$+$	undef	$-$		$-$	undef	$+$
f	↗	undef	↗	max	↘	undef	↘
	⌣		⌢		⌢		⌣

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = xe^{-x^2/2}$

Solution

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = xe^{-x^2/2}$

Solution

Step 1: *Find the domain of f.*

$$f(x) = xe^{-x^2/2} \quad \textit{is defined for all real } x.$$

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = xe^{-x^2/2}$

Solution

Step 1: *Domain of f is $(-\infty, \infty)$*

Step 2: *Find asymptotes. Horizontal asymptote: $y = 0$ since*

$$\lim_{x \to \pm\infty} xe^{-x^2/2} = 0 \quad \textit{(exponential dominates)}$$

No vertical asymptotes.

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = xe^{-x^2/2}$

Solution

Step 1: *Domain of f is $(-\infty, \infty)$*

Step 2: *Horizontal asymptote: $y = 0$*

Step 3: *Check symmetry.*

$$f(-x) = (-x)e^{-(-x)^2/2} = -xe^{-x^2/2} = -f(x)$$

Thus, f is an odd function (symmetric about origin).

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = xe^{-x^2/2}$

Solution

Step 1: *Domain of f is* $(-\infty, \infty)$

Step 2: *Horizontal asymptote:* $y = 0$

Step 3: *Odd function.*

Step 4: *Find intercepts.* y*-intercept:* $f(0) = 0$. x*-intercept:* $x = 0$.

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = xe^{-x^2/2}$

Solution

Step 1: *Domain of f is* $(-\infty, \infty)$

Step 2: *Horizontal asymptote:* $y = 0$

Step 3: *Odd function.*

Step 4: *Intercept:* $(0, 0)$.

Step 5: *Examine* $f'(x)$.

$$f'(x) = e^{-x^2/2}(1 - x^2) = 0$$

$$\Rightarrow x = \pm 1$$

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Sketch the graph of $y = xe^{-x^2/2}$

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$$\Rightarrow x = \pm 1$$

Step 6: *Examine* $f''(x)$.

$$f''(x) = xe^{-x^2/2}(x^2 - 3) = 0$$

$$\Rightarrow x = 0, \pm\sqrt{3}$$

Sketching the Graph of a Function: Examples

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			CP				CP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
y'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
y''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
y	$\searrow$		$\searrow$	min	$\nearrow$		$\nearrow$	max	$\searrow$		$\searrow$
	$\frown$	infl	$\smile$		$\smile$	infl	$\frown$		$\frown$	infl	$\smile$

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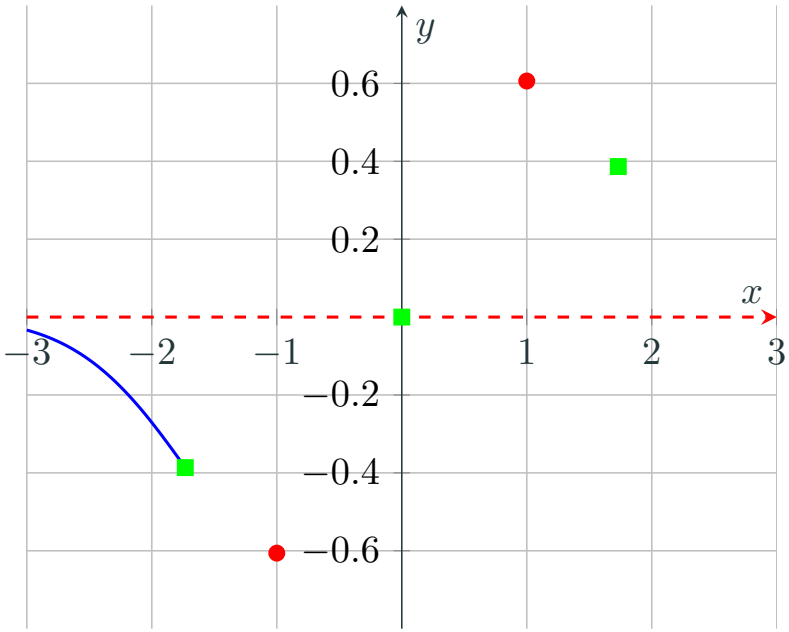

				CP				CP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
y'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
y''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
y	↘		↘	min	↗		↗	max	↘		↘
	⌒	infl	⌣		⌣	infl	⌒		⌒	infl	⌣

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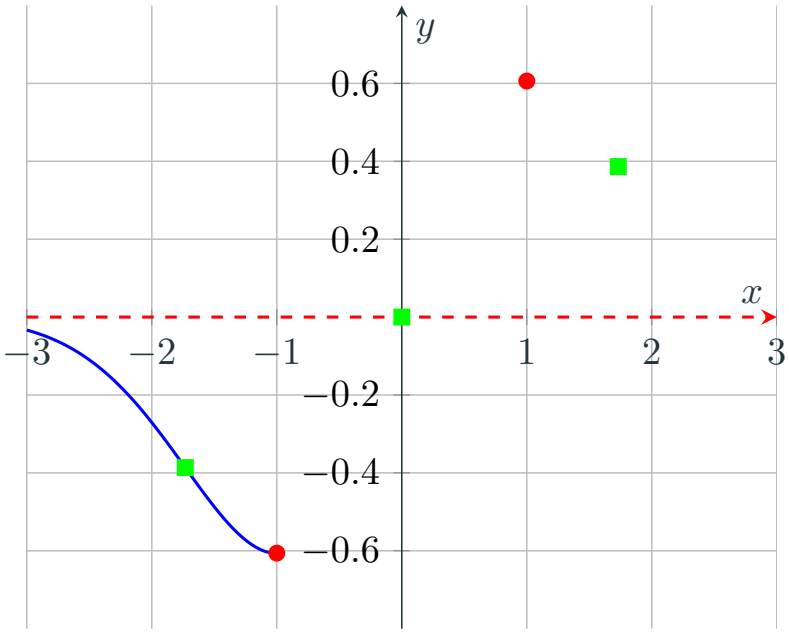

				CP				CP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
y'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
y''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
y	$\searrow$		$\searrow$	min	$\nearrow$		$\nearrow$	max	$\searrow$		$\searrow$
	$\frown$	infl	$\smile$		$\smile$	infl	$\frown$		$\frown$	infl	$\smile$

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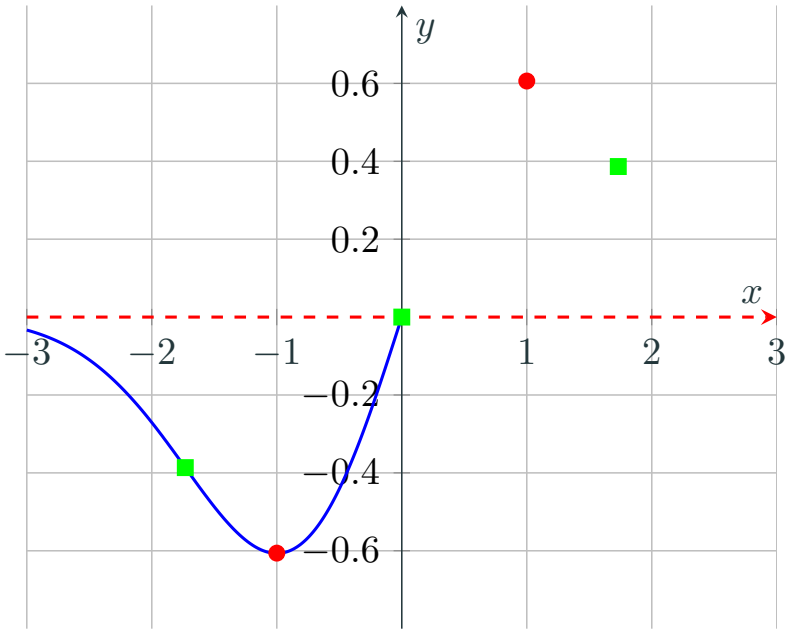

				CP				CP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
y'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
y''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
y	$\searrow$		$\searrow$	min	$\nearrow$		$\nearrow$	max	$\searrow$		$\searrow$
	$\frown$	infl	$\smile$		$\smile$	infl	$\frown$		$\frown$	infl	$\smile$

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Sketch the graph of $y = xe^{-x^2/2}$

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Step 1: *Domain of f is* $(-\infty, \infty)$

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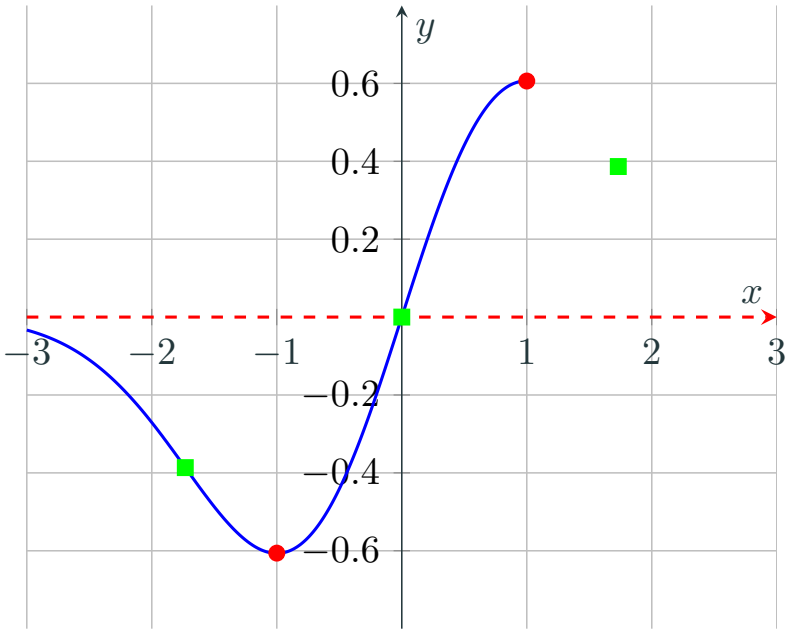

				CP				CP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
y'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
y''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
y	$\searrow$		$\searrow$	min	$\nearrow$		$\nearrow$	max	$\searrow$		$\searrow$
	$\frown$	infl	$\smile$		$\smile$	infl	$\frown$		$\frown$	infl	$\smile$

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $y = xe^{-x^2/2}$

Solution

Step 1: *Domain of f is* $(-\infty, \infty)$

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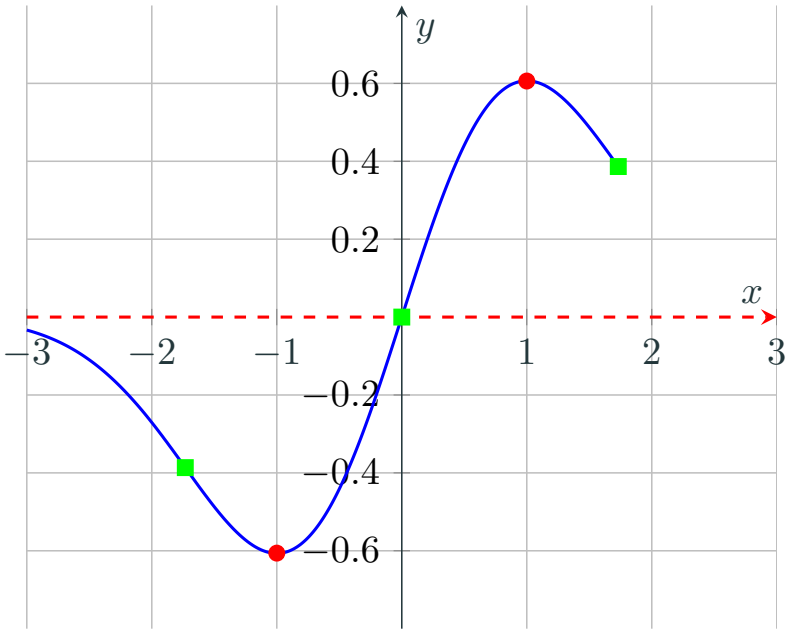

				CP				CP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
y'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
y''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
y	$\searrow$		$\searrow$	min	$\nearrow$		$\nearrow$	max	$\searrow$		$\searrow$
	$\frown$	infl	$\smile$		$\smile$	infl	$\frown$		$\frown$	infl	$\smile$

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Sketch the graph of $y = xe^{-x^2/2}$

Solution

Step 1: *Domain of f is* $(-\infty, \infty)$

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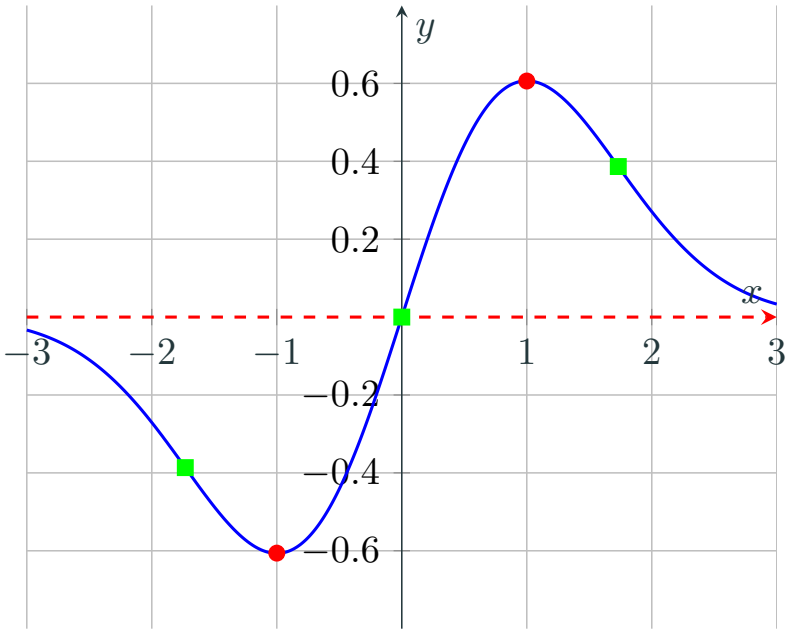
Step 4: *Intercept:* $(0, 0)$.



				CP				CP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
y'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
y''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
y	$\searrow$		$\searrow$	min	$\nearrow$		$\nearrow$	max	$\searrow$		$\searrow$
	$\frown$	infl	$\smile$		$\smile$	infl	$\frown$		$\frown$	infl	$\smile$

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $f(x) = (x^2 - 1)^{2/3}$

Solution

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $f(x) = (x^2 - 1)^{2/3}$

Solution

Step 1: *Find the domain of f.*

$$f(x) = (x^2 - 1)^{2/3} \quad \textit{is defined for all real } x.$$

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $f(x) = (x^2 - 1)^{2/3}$

Solution

Step 1: *Domain of f is $(-\infty, \infty)$*

Step 2: *Find asymptotes. No asymptotes since*

$$\lim_{x \to \pm\infty} (x^2 - 1)^{2/3} = +\infty$$

The function grows without bound as $|x| \to \infty$.

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $f(x) = (x^2 - 1)^{2/3}$

Solution

Step 1: *Domain of f is $(-\infty, \infty)$*

Step 2: *No asymptotes*

Step 3: *Check symmetry.*

$$f(-x) = ((-x)^2 - 1)^{2/3} = (x^2 - 1)^{2/3} = f(x)$$

Thus, f is an even function (symmetric about y-axis).

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $f(x) = (x^2 - 1)^{2/3}$

Solution

Step 1: *Domain of f is* $(-\infty, \infty)$

Step 2: *No asymptotes*

Step 3: *Even function.*

Step 4: *Find intercepts. y-intercept:* $f(0) = 1$. *x-intercepts:* $f(\pm 1) = 0$.

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $f(x) = (x^2 - 1)^{2/3}$

Solution

Step 1: *Domain of* f *is* $(-\infty, \infty)$

Step 2: *No asymptotes*

Step 3: *Even function.*

Step 4: *Intercepts:* $(0, 1)$, x*-intercepts* $(\pm 1, 0)$.

Step 5: *Examine* $f'(x)$.

$f'(x) = \frac{4x}{3}(x^2 - 1)^{-1/3} = 0$

$\Rightarrow x = 0$ *(CP).* f *has singular points at* $x = \pm 1$.

Sketching the Graph of a Function: Examples

Example

Sketch the graph of $f(x) = (x^2 - 1)^{2/3}$

Solution

Step 1: *Domain of f is* $(-\infty, \infty)$

Step 2: *No asymptotes*

Step 3: *Even function.*

Step 4: *Intercepts:* $(0, 1)$, *x-intercepts* $(\pm 1, 0)$.

Step 5: *Examine* $f'(x)$.

$f'(x) = \frac{4x}{3}(x^2 - 1)^{-1/3} = 0$

$\Rightarrow x = 0$ *(CP). f has singular points at* $x = \pm 1$.

Step 6: *Examine* $f''(x)$.

$$f''(x) = \frac{4}{9}(x^2 - 1)^{-4/3}(x^2 - 3)$$

$f''(x) = 0 \iff x = \pm\sqrt{3}$.

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				SP		CP		SP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$-$		$-$	undef	$+$	0	$-$	undef	$+$		$+$
f''	$+$	0	$-$	undef	$-$		$-$	undef	$-$	0	$+$
f	$\searrow$		$\searrow$	min	$\nearrow$	max	$\searrow$	min	$\nearrow$		$\nearrow$
	$\smile$	infl	$\frown$		$\frown$		$\frown$		$\frown$	infl	$\smile$

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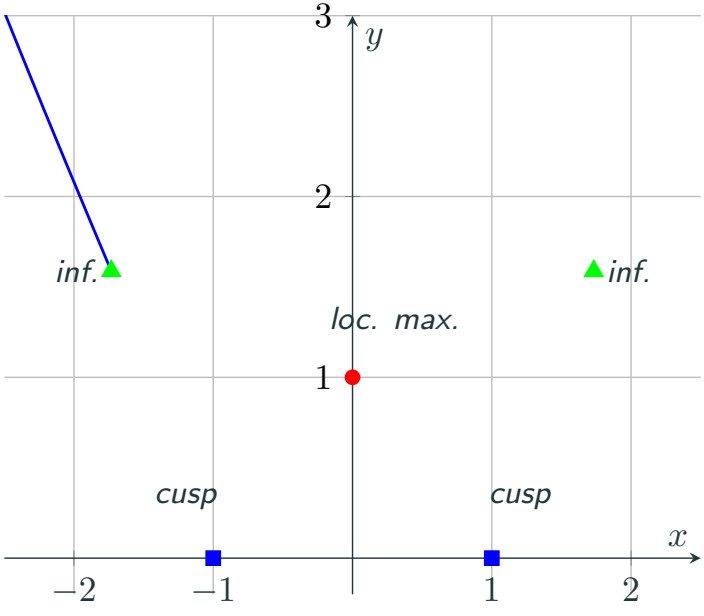

				SP		CP		SP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$-$		$-$	undef	$+$	0	$-$	undef	$+$		$+$
f''	$+$	0	$-$	undef	$-$		$-$	undef	$-$	0	$+$
f	$\searrow$		$\searrow$	min	$\nearrow$	max	$\searrow$	min	$\nearrow$		$\nearrow$
	$\smile$	infl	$\frown$		$\frown$		$\frown$		$\frown$	infl	$\smile$

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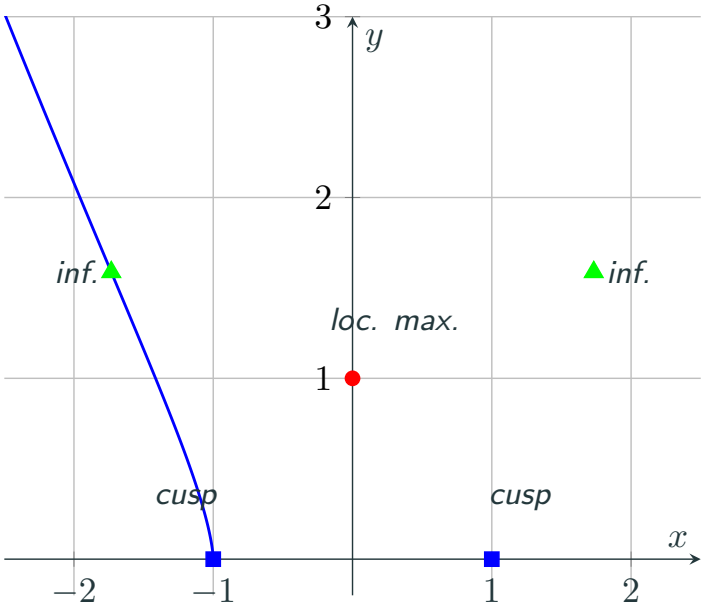

				SP		CP		SP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$-$		$-$	undef	$+$	0	$-$	undef	$+$		$+$
f''	$+$	0	$-$	undef	$-$		$-$	undef	$-$	0	$+$
f	↘		↘	min	↗	max	↘	min	↗		↗
	⌣	infl	⌢		⌢		⌢		⌢	infl	⌣

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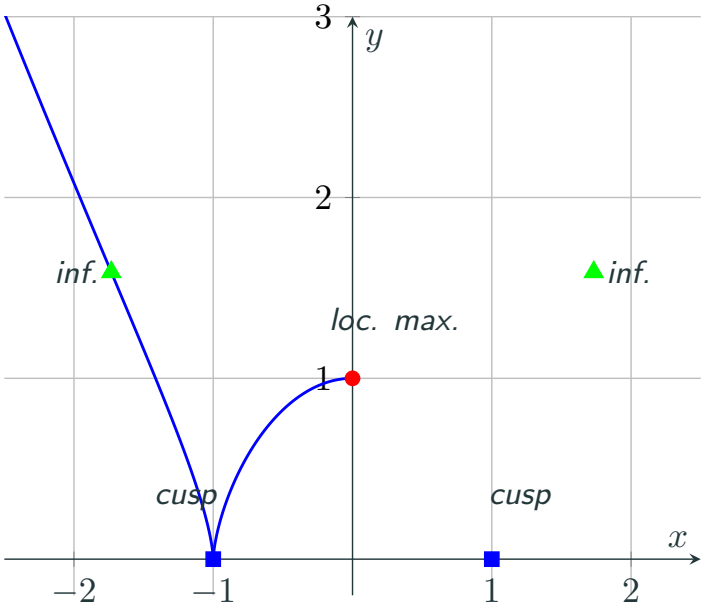

			SP		CP		SP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$-$		$-$	undef	$+$	0	$-$	undef	$+$		$+$
f''	$+$	0	$-$	undef	$-$		$-$	undef	$-$	0	$+$
f	↘		↘	min	↗	max	↘	min	↗		↗
	⌣	infl	⌢		⌢		⌢		⌢	infl	⌣

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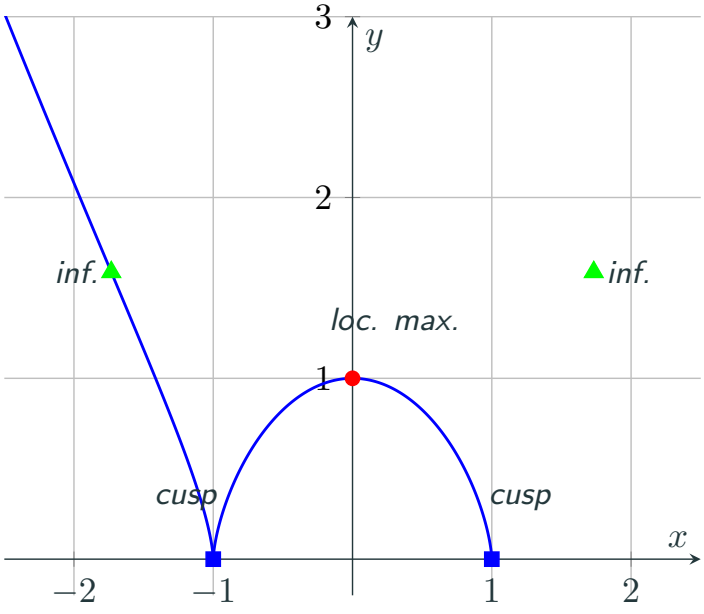

				SP		CP		SP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$-$		$-$	undef	$+$	0	$-$	undef	$+$		$+$
f''	$+$	0	$-$	undef	$-$		$-$	undef	$-$	0	$+$
f	↘		↘	min	↗	max	↘	min	↗		↗
	⌣	infl	⌢		⌢		⌢		⌢	infl	⌣

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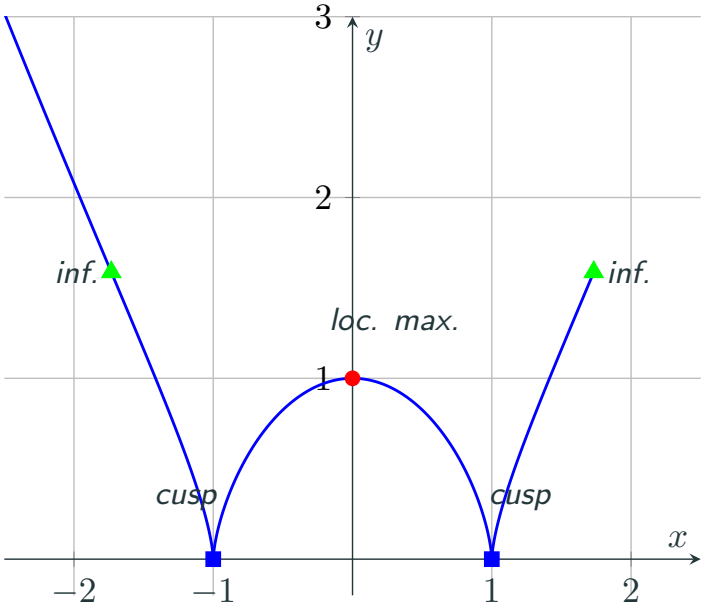

				SP		CP		SP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$-$		$-$	undef	$+$	0	$-$	undef	$+$		$+$
f''	$+$	0	$-$	undef	$-$		$-$	undef	$-$	0	$+$
f	↘		↘	min	↗	max	↘	min	↗		↗
	⌣	infl	⌢		⌢		⌢		⌢	infl	⌣

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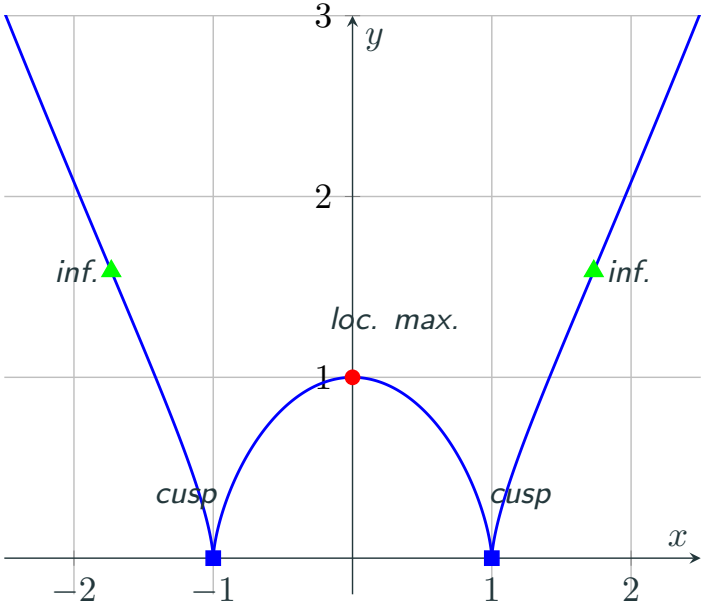

				SP		CP		SP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$-$		$-$	undef	$+$	0	$-$	undef	$+$		$+$
f''	$+$	0	$-$	undef	$-$		$-$	undef	$-$	0	$+$
f	↘		↘	min	↗	max	↘	min	↗		↗
	⌣	infl	⌢		⌢		⌢		⌢	infl	⌣