MAT123 MATHEMATICS I

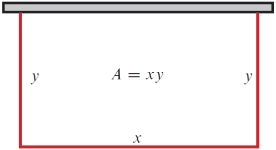
Lecture 15: Applications of Differentiation (Continued), Integration

Outline

Applications of Differentiation

Example. A rectangular animal enclosure is to be constructed having one side along an existing long wall and the other three sides fenced. If 100 m of fence are available, what is the largest possible area for the enclosure?

Solution.



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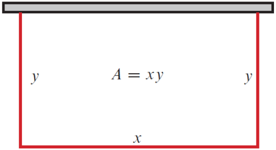

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Setting $x = 100 - 2y$, the area A can be expressed as a function of y

$$A = A(y) = (100 - 2y) \cdot y = 100y - 2y^2.$$

on the interval $[0, 50]$.

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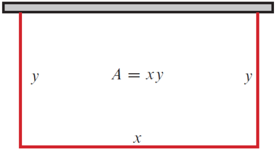

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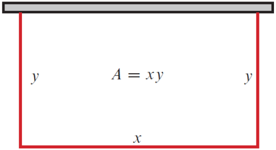

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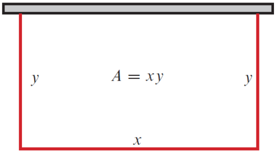

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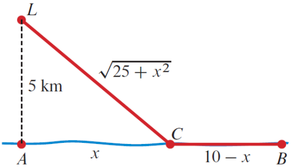
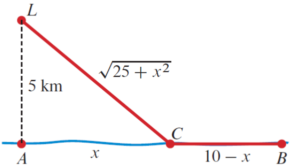
Evaluating the area at the critical point and the endpoints, we find
$A(0) = 0, \quad A(25) = 1250, \quad A(50) = 0.$
Thus, the maximum area is $1250\,\text{m}^2$.

Example. A lighthouse L is located on a small island 5 km north of a point A on a straight east-west shoreline. A cable is to be laid from L to point B on the shoreline 10 km east of A. The cable will be laid through the water in a straight line from L to a point C on the shoreline between A and B, and from there to B along the shoreline.



Example. The part of the cable lying in the water costs \$5,000/km, and the part along the shoreline costs \$3,000/km.

(a) Where should C be chosen to minimize the total cost of the cable?



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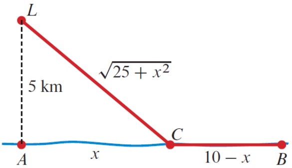

Let x be the distance from A to C.

- The total cost T of the cable can be expressed as $T = 5000 \cdot \sqrt{5^2 + x^2} + 3000 \cdot (10 - x)$ on the interval $[0, 10]$.

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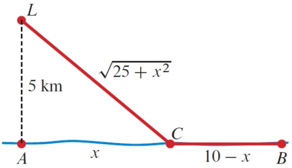

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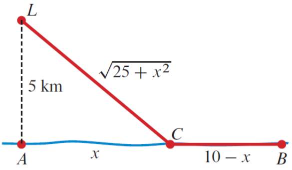

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- Differentiating, we get $T'(x) = \frac{5000 \cdot x}{\sqrt{25+x^2}} - 3000 = 0$
- Solving for x, we find $x = \frac{3000 \cdot \sqrt{25+x^2}}{5000} \Rightarrow 5000x = 3000\sqrt{25 + x^2}$ $\Rightarrow 25x^2 = 9(25 + x^2) \Rightarrow x^2 = \frac{225}{16} \Rightarrow x = \frac{15}{4}$

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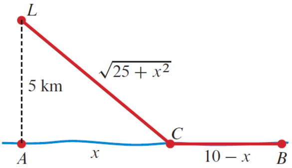

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- Since $T(0) = 55,000$, $T(10) \approx 55,902$, and $T(\frac{15}{4}) = 50,000$,

the minimum cost occurs at $x = \frac{15}{4} = 3.75$ km from A.

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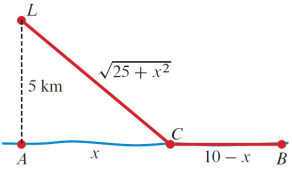
Example. The part of the cable lying in the water costs \$5,000/km, and the part along the shoreline costs \$3,000/km.

(b) Where should C be chosen if B is only 3 km from A?



Let x be the distance from A to C.

- The total cost T of the cable can be expressed as $T = 5000 \cdot \sqrt{5^2 + x^2} + 3000 \cdot (3 - x)$ on the interval $[0, 3]$.
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- Since $3.75 \notin [0, 3]$, and $T(0) = 34,000$, $T(3) \approx 29,155$,

the minimum cost occurs at $x = 3$ km from A, i.e.,
to minimize the total cost, the cable should go straight from L to B.

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6. Find the required extreme value of the function Q considering any critical points, singular points, and endpoints.

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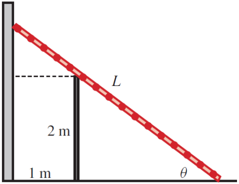
6. Find the required extreme value of the function Q considering any critical points, singular points, and endpoints.
7. Check the solution and make a concluding statement.

Example. Find the length of the shortest ladder that can extend from a vertical wall, over a fence 2 m high located 1 m away from the wall, to a point on the ground outside the fence.



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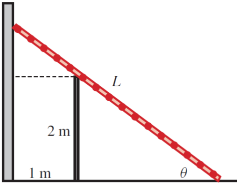

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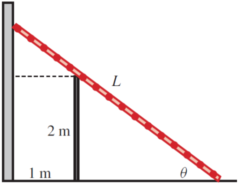

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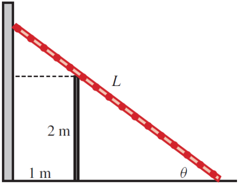

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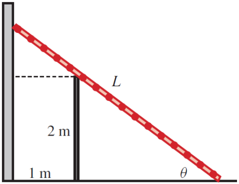

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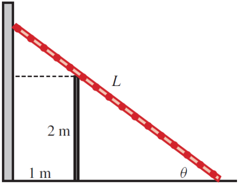

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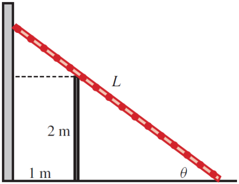

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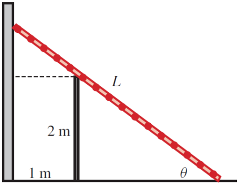

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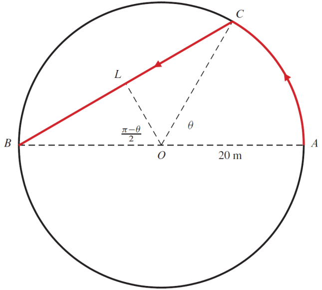
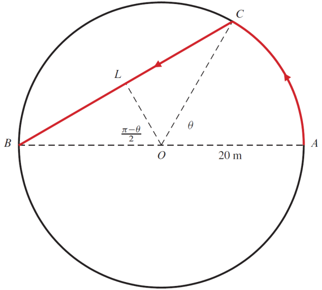
- $\sin\theta = \dfrac{\tan\theta}{\sec\theta} = \dfrac{2^{1/3}}{\sqrt{1+2^{2/3}}}$
- Therefore, the minimum value of $L(\theta)$ is

$$\frac{1}{\cos\theta} + \frac{2}{\sin\theta} = (1+2^{2/3})^{1/2} + 2\frac{(1+2^{2/3})^{1/2}}{2^{1/3}}$$
$$= (1+2^{1/3})^{3/2} \approx 4.16.$$

Example. A man can run twice as fast as he can swim. He is standing at point A on the edge of a circular swimming pool 40 m in diameter, and he wishes to get to the diametrically opposite point B as quickly as possible.



Example. He can run around the edge to point C, then swim directly from C to B. Where should C be chosen to minimize the total time taken to get from A to B?



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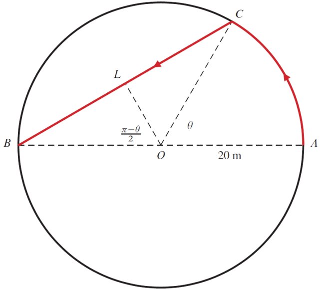

Suppose the man swims at a rate of k m/s.

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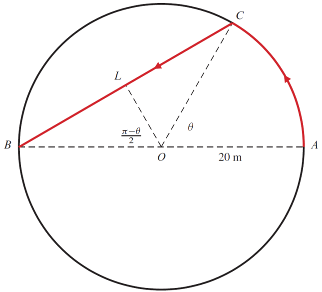

Suppose the man swims at a rate of k m/s.
Then the total time is
$t = \frac{20\theta}{2k} + \frac{40}{k}\sin\frac{\pi-\theta}{2}$
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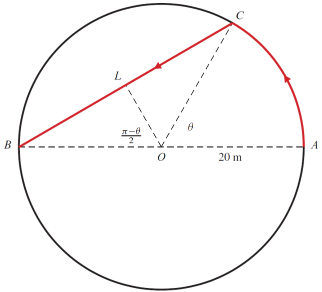

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- Differentiating $t(\theta)$ and setting it to zero:

$$t'(\theta) = \frac{10}{k} - \frac{20}{k}\cos\frac{\pi-\theta}{2} = 0 \Rightarrow$$

$$\frac{10}{k} - \frac{20}{k}\cos\frac{\pi-\theta}{2} = 0 \Rightarrow \cos\frac{\pi-\theta}{2} = \frac{1}{2} \Rightarrow$$

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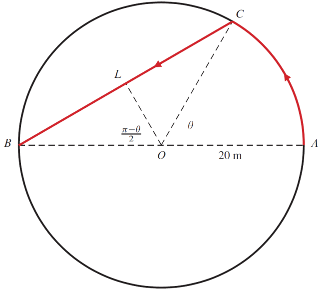

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$$\frac{\pi-\theta}{2} = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}.$$

- Evaluating $t(\theta)$ at this critical point, as well as the endpoints:

$$t(0) = \frac{40}{k},\ t(\frac{\pi}{3}) = \frac{10\pi}{3k} + \frac{40\sqrt{3}}{2k} \approx \frac{45.11}{k},$$

$$t(\pi) = \frac{10\pi}{k} \approx \frac{31.4}{k}.$$

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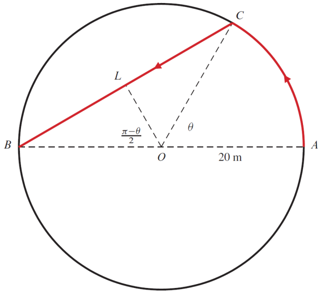

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on $[0, \pi]$.

- Differentiating $t(\theta)$ and setting it to zero:

$$t'(\theta) = \frac{10}{k} - \frac{20}{k}\cos\frac{\pi-\theta}{2} = 0 \Rightarrow$$

$$\frac{10}{k} - \frac{20}{k}\cos\frac{\pi-\theta}{2} = 0 \Rightarrow \cos\frac{\pi-\theta}{2} = \frac{1}{2} \Rightarrow$$

$$\frac{\pi-\theta}{2} = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}.$$

- Evaluating $t(\theta)$ at this critical point, as well as the endpoints:

$$t(0) = \frac{40}{k},\ t(\frac{\pi}{3}) = \frac{10\pi}{3k} + \frac{40\sqrt{3}}{2k} \approx \frac{45.11}{k},$$

$$t(\pi) = \frac{10\pi}{k} \approx \frac{31.4}{k}.$$

Integration

Sums and Sigma Notation

Sigma notation

If m and n are integers with $m \leq n$, and if f is a function defined at the integers $m, m+1, m+2, \ldots, n$, the symbol $\sum_{i=m}^{n} f(i)$ represents the sum of the values of f at those integers:

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n)$$

The explicit sum appearing on the right side of this equation is the **expansion** of the sum represented in sigma notation on the left side.

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The explicit sum appearing on the right side of this equation is the **expansion** of the sum represented in sigma notation on the left side.

Example. $$\sum_{i=1}^{5} f(i) = \sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

Sums and Sigma Notation

Sometimes we use a subscripted variable a_i to denote the ith term of a general sum instead of using the functional notation $f(i)$. In this case, we write

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- $\displaystyle\sum_{i=m}^{n} \big(Af(i) + Bg(i)\big) = A\sum_{i=m}^{n} f(i) + B\sum_{i=m}^{n} g(i)$
- $\displaystyle\sum_{j=m}^{m+n} f(j) = \sum_{i=0}^{n} f(i+m).$

Sums and Sigma Notation

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$$S = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

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$$\begin{array}{ccccccccccc} S & = & 1 & + & 2 & + & 3 & + \cdots + & (n-1) & + & n \\ S & = & n & + & (n-1) & + & (n-2) & + \cdots + & 2 & + & 1 \\ \hline \end{array}$$

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Dividing both sides by 2 gives us the formula for the sum of the first n integers:

$$S = \frac{n(n+1)}{2}.$$

Sums and Sigma Notation

Theorem *(Summation formulas)*

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(c) $\displaystyle\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$

(d) $\displaystyle\sum_{i=1}^{n} r^{i-1} = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$, for $r \neq 1$.

Sums and Sigma Notation

Let us prove the equality given in (c):

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof. To prove (c), we write n copies of the identity $(k+1)^3 - k^3 = 3k^2 + 3k + 1$, one for each value of k from 1 to n, *and add them up:*

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4^3 & - & 3^3 & = & 3\times 3^2 & + & 3\times 3 & + & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
n^3 & - & (n-1)^3 & = & 3(n-1)^2 & + & 3(n-1) & + & 1 \\
(n+1)^3 & - & n^3 & = & 3n^2 & + & 3n & + & 1
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\cancel{4^3} & - & \cancel{3^3} & = & 3\times 3^2 & + & 3\times 3 & + & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\cancel{n^3} & - & \cancel{(n-1)^3} & = & 3(n-1)^2 & + & 3(n-1) & + & 1 \\
(n+1)^3 & - & \cancel{n^3} & = & 3n^2 & + & 3n & + & 1 \\
\hline
(n+1)^3 & - & 1^3 & = & 3\left(\sum_{i=1}^{n} i^2\right) & + & 3\left(\sum_{i=1}^{n} i\right) & + & n \\
 & & & = & 3\left(\sum_{i=1}^{n} i^2\right) & + & \dfrac{3n(n+1)}{2} & + & n.
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Example. Evaluate $\sum_{k=m+1}^{n}(6k^2 - 4k + 3)$, where $1 \leq m < n$.

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$$\sum_{k=1}^{n}(6k^2 - 4k + 3) = 6\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 3$$

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Now we can find the value of the sum from $m+1$ to n:

$$\sum_{k=m+1}^{n}(6k^2-4k+3)=\sum_{k=1}^{n}(6k^2-4k+3)-\sum_{k=1}^{m}(6k^2-4k+3)$$

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Now we can find the value of the sum from $m+1$ to n:

$$\begin{aligned}\sum_{k=m+1}^{n}(6k^2-4k+3) &= \sum_{k=1}^{n}(6k^2-4k+3) - \sum_{k=1}^{m}(6k^2-4k+3)\\ &= 2n^3+n^2+2n-\left(2m^3+m^2+2m\right).\end{aligned}$$