MAT123 MATHEMATICS I

Lecture 16: Integration (Continued)

Outline

Integration

Areas as Limits of Sums

(i) The area of a plane region is a nonnegative real number of *square units*.

Areas as Limits of Sums

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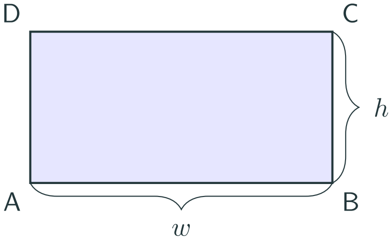
(ii) The area of a rectangle with width w and height h is $A = wh$.



Area $ABCD = wh$

Areas as Limits of Sums

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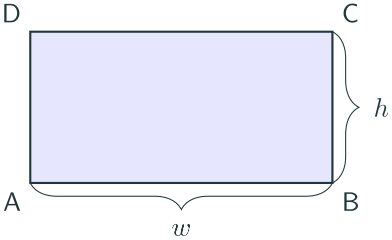
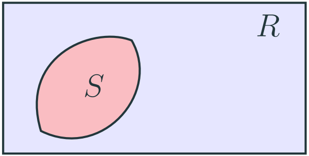


Area $ABCD = wh$

(iii) The areas of congruent plane regions are equal.

Areas as Limits of Sums

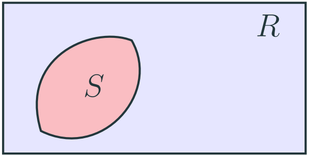
(iv) If region S is contained in region R, then the area of S is less than or equal to that of R.



Area $S <$ Area R

Areas as Limits of Sums

(iv) If region S is contained in region R, then the area of S is less than or equal to that of R.



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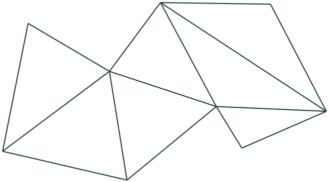
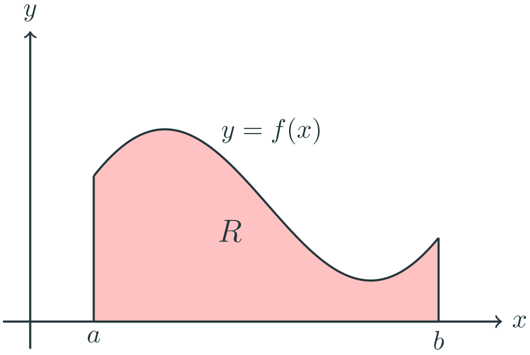


area of polygon $=$
sum of areas of triangles

(v) If region R is a union of (finitely many) nonoverlapping regions, then the area of R is the sum of the areas of those regions.

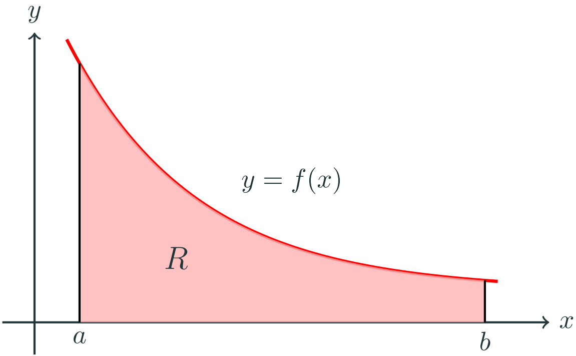
Areas as Limits of Sums



The basic area problem: How can we find the area of region R?

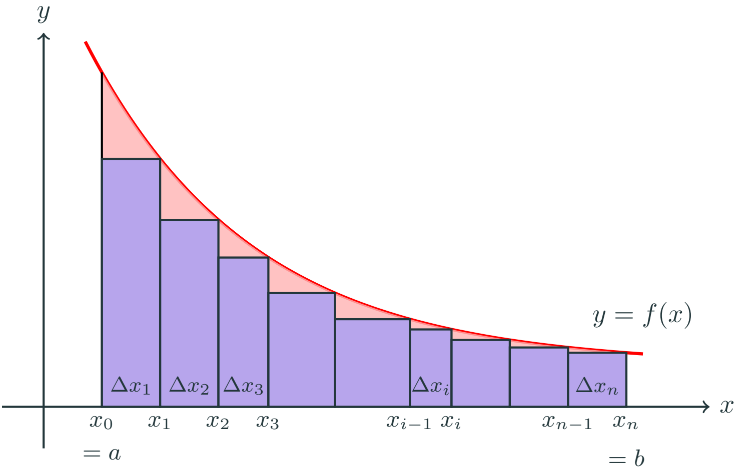
Areas as Limits of Sums

Consider the following region R in the plane:



The Idea: Approximating the area of R using the area of a very-well known geometric object, the rectangle.

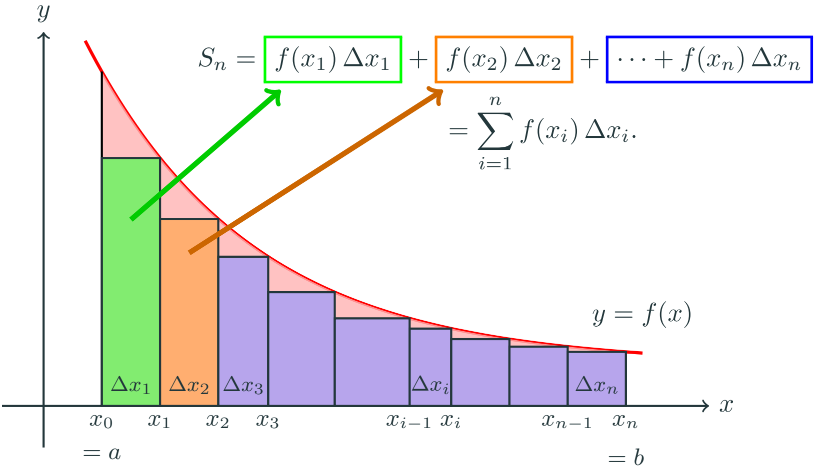
Areas as Limits of Sums



$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$$

$$\Delta x_i = x_i - x_{i-1}, \quad (i = 1, 2, 3, \ldots, n).$$

Areas as Limits of Sums



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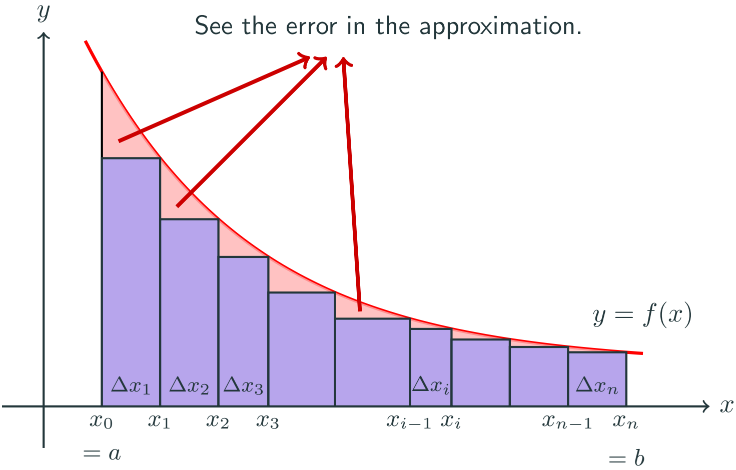
Areas as Limits of Sums

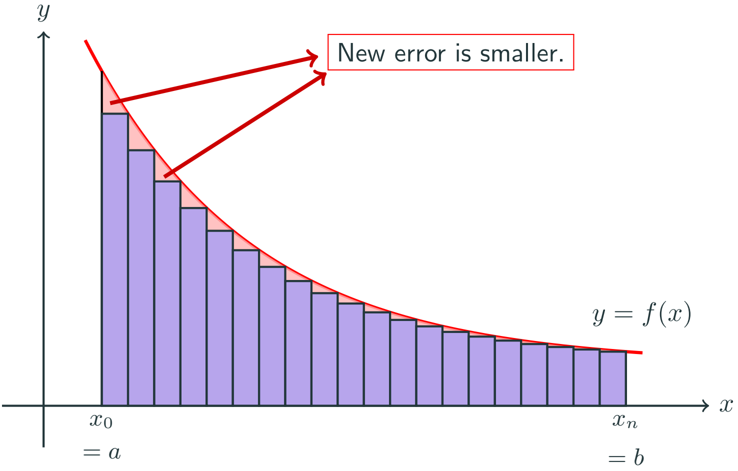


$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$$

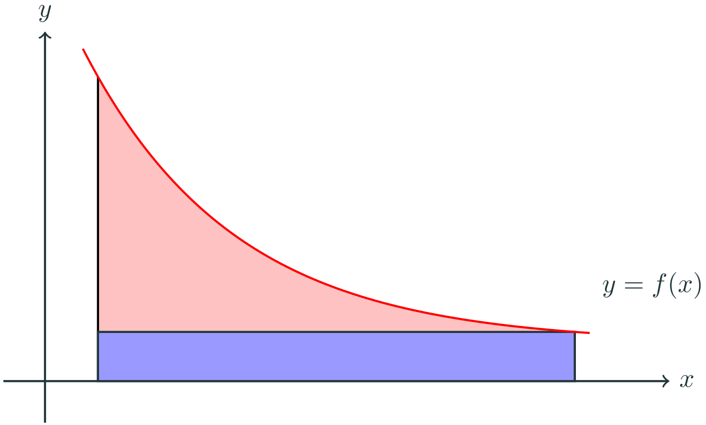
$$\Delta x_i = x_i - x_{i-1}, \quad (i = 1, 2, 3, \ldots, n).$$

Areas as Limits of Sums

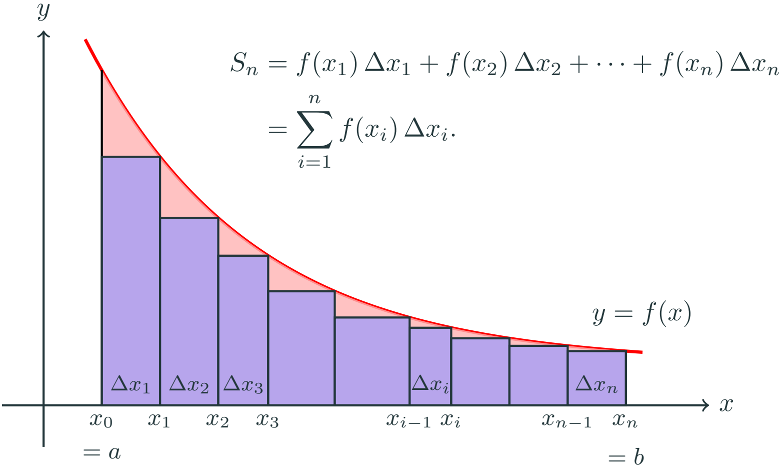
We can use more rectangles to approximate the area better.



Eventually, we can make the rectangles so narrow that the error in the approximation becomes negligible.



Areas as Limits of Sums



Area of $R = \lim_{\substack{n\to\infty \\ \max \Delta x_i \to 0}} S_n$.

Example. Find the area A of the region lying under the straight line $y = x + 1$, above the x-axis, and between the lines $x = 0$ and $x = 2$.

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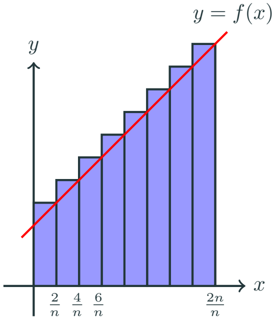

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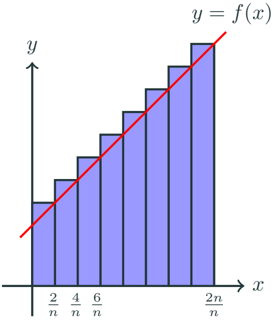

$$
\begin{aligned}
S_n &= \sum_{i=1}^{n} \left(\frac{2i}{n} + 1 \right) \frac{2}{n} \\
&= \left(\frac{2}{n} \right) \left[\frac{2}{n} \sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \right] \\
&= \left(\frac{2}{n} \right) \left[\frac{2}{n} \frac{n(n+1)}{2} + n \right] \\
&= 2 \frac{n+1}{n} + 2.
\end{aligned}
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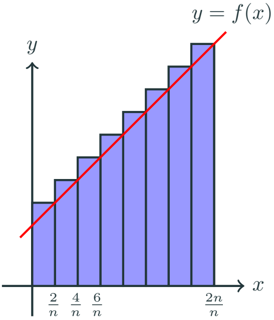

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$$

Therefore, $A = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(2 \frac{n+1}{n} + 2 \right) = 2 + 2 = 4.$

Areas as Limits of Sums

Example. Find the area of the region bounded by the parabola $y = x^2$ and the straight lines $y = 0$, $x = 0$, and $x = b$, where $b > 0$.

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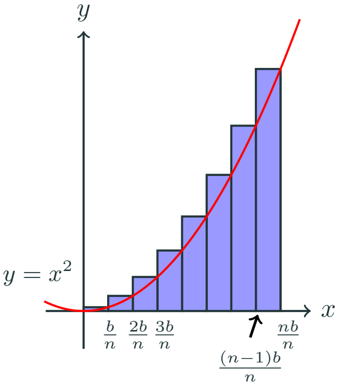
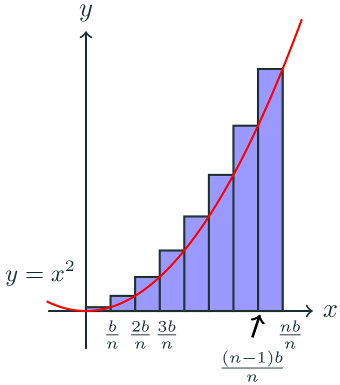


Areas as Limits of Sums

Example. Find the area of the region bounded by the parabola $y = x^2$ and the straight lines $y = 0$, $x = 0$, and $x = b$, where $b > 0$.

Solution.

$$S_n = \sum_{i=1}^{n} \left(\frac{ib}{n}\right)^2 \frac{b}{n}$$
$$= \frac{b^3}{n^3} \sum_{i=1}^{n} i^2$$
$$= \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6}.$$



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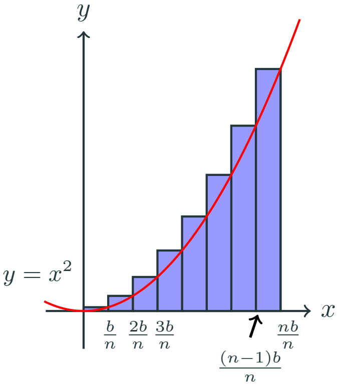

Hence, the required area is

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} b^3 \frac{(n+1)(2n+1)}{6n^2} = \frac{b^3}{3}.$$

Let P be a finite set of points arranged in order between a and b on the real line, say

$$P = \{x_0, x_1, x_2, x_3, \ldots, x_{n-1}, x_n\},$$

where $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$. Such a set P is called a *partition* of the interval $[a, b]$.

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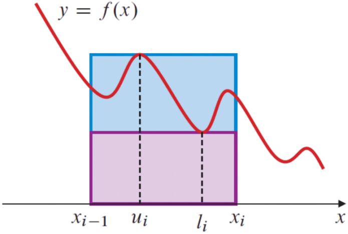
$$\Delta x_i = x_i - x_{i-1}, \quad \text{(for } 1 \leq i \leq n\text{)}$$

and we call the greates of these numbers Δx_i the *norm* of the partition P, denoted by $\|P\|$:

$$\|P\| = \max_{1 \leq i \leq n} \Delta x_i.$$

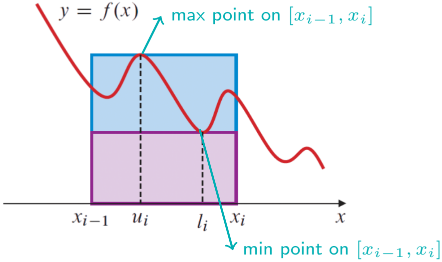
The Definite Integral

Riemann Sums



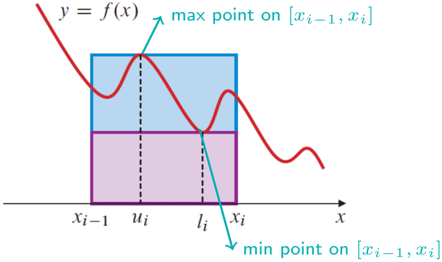
The Definite Integral

Riemann Sums



If A_i is that part of the area under $y = f(x)$ and above the x-axis that lies in the vertical strip between $x = x_{i-1}$ and $x = x_i$, then

$$f(l_i)\Delta x_i \leq A_i \leq f(u_i)\Delta x_i.$$



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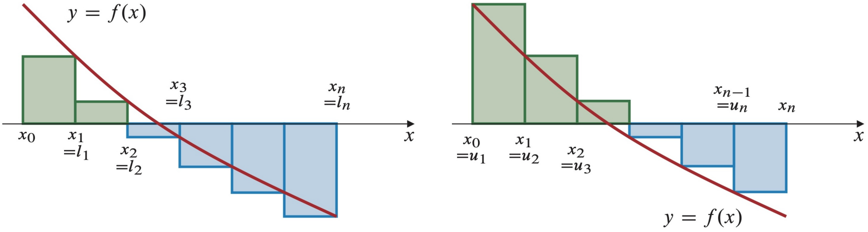
The *lower (Riemann) sum*, denoted by $L(P, f)$ and the *upper (Riemann) sum*, denoted by $U(P, f)$, for the function f and the partition P are defined as follows:

$$L(f, P) = f(l_1)\Delta x_1 + f(l_2)\Delta x_2 + \cdots + f(l_n)\Delta x_n = \sum_{i=1} f(l_i)\Delta x_i$$

$$U(f, P) = f(u_1)\Delta x_1 + f(u_2)\Delta x_2 + \cdots + f(u_n)\Delta x_n = \sum_{i=1} f(u_i)\Delta x_i.$$

The Definite Integral

Partitions



A lower Riemann sum and an upper Riemann sum for a decreasing function f.

Example. Calculate the lower and upper Riemann sums for the function $f(x) = x^2$ on the interval $[0, a]$ (where $a > 0$), corresponding to the partition P_n of $[0, a]$ into n subintervals of equal length.

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Solution. Each subinterval of P_n has length $\Delta x = a/n$, and the division points are given by $x_i = ia/n$, for $i = 0, 1, 2, \ldots, n$. Since x^2 is increasing on $[0, a]$, its minimum and maximum values over the ith subinterval $[x_{i-1}, x_i]$ occur at $l_i = x_{i-1}$ and $u_i = x_i$, respectively.

$$L(f, P_n) = \sum_{i=1}^{n} (x_{i-1})^2 \Delta x = \frac{a^3}{n^3} \sum_{i=1}^{n} (i-1)^2$$

$$= \frac{a^3}{n^3} \sum_{j=0}^{n-1} j^2 = \frac{a^3}{n^3} \frac{(n-1)n(2(n-1)+1)}{6} = \frac{(n-1)(2n-1)a^3}{6n^2},$$

$$
\begin{aligned}
L(f, P_n) &= \sum_{i=1}^{n} (x_{i-1})^2 \Delta x = \frac{a^3}{n^3} \sum_{i=1}^{n} (i-1)^2 \\
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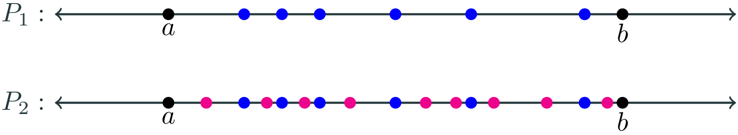
U(f, P_n) &= \sum_{i=1}^{n} (x_i)^2 \Delta x = \frac{a^3}{n^3} \sum_{i=1}^{n} i^2 \\
&= \frac{a^3}{n^3} \sum_{j=1}^{n} j^2 = \frac{a^3}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{(2n+1)(n+1)a^3}{6n^2}.
\end{aligned}
$$

The Definite Integral

If P_1 and P_2 are two partitions of $[a, b]$ such that every point of P_1 also belongs to P_2, then we say that P_2 is a *refinement* of P_1.



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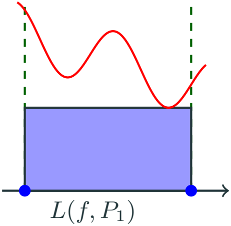
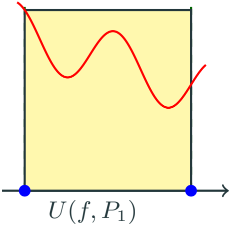
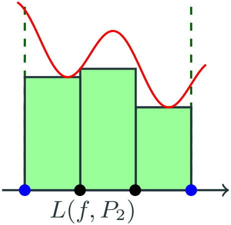
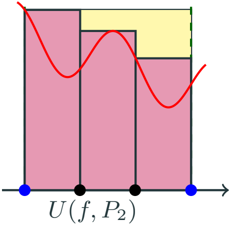
The Definite Integral









The Definite Integral

If P_2 is a refinement of P_1, then we have

$$L(f, P_1) \leq L(f, P_2) \leq U(f, P_2) \leq U(f, P_1).$$

The Definite Integral

If P_2 is a refinement of P_1, then we have

$$L(f, P_1) \leq L(f, P_2) \leq U(f, P_2) \leq U(f, P_1).$$

Also, given any two partitions P_1 and P_2, we can have a common refinement P by combining these partitions. So

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2).$$

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$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2).$$

Hence, every lower sum is less than or equal to every upper sum!

The Definite Integral

Therefore, there is at least one number I such that

$$L(f, P) \leq I \leq U(f, P)$$

for every partition P of $[a, b]$. (By the completeness property of the real numbers! That is there are no "gaps" or "missing points" in the real number line.)

The Definite Integral

Definition. *The definite integral*

Suppose there is exactly one number I such that for every partition P of $[a, b]$, we have

$$L(f, P) \leq I \leq U(f, P).$$

Then we say that f is *integrable* on $[a, b]$ and we define the *definite integral* of f on $[a, b]$ to be the number I. The definite integral is denoted by the symbol

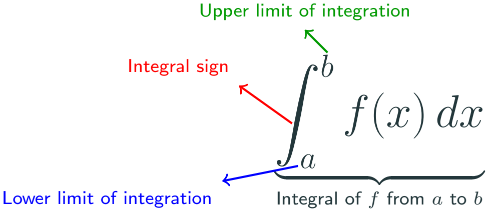
$$\int_a^b f(x)\,dx = I.$$

The Definite Integral

The definite integral of $f(x)$ over $[a, b]$ is a *number*; it is not a function of x. So

$$\int_a^b f(x)dx = \int_a^b f(t)dt.$$

The Definite Integral



The Definite Integral

Upper limit of integration

The function is the integrand

Integral sign

x is the variable of integration

$$\int_a^b f(x)\,dx$$

Lower limit of integration

Integral of f from a to b

Example. Show that $f(x) = x^2$ is integrable over the interval $[0, a]$, where $a > 0$, and evaluate $\int_0^a x^2 dx$.

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Solution.

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} \frac{(n-1)(2n-1)a^3}{6n^2} = \frac{a^3}{3},$$

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{(2n+1)(n+1)a^3}{6n^2} = \frac{a^3}{3}.$$

Example. Show that $f(x) = x^2$ is integrable over the interval $[0, a]$, where $a > 0$, and evaluate $\int_0^a x^2 dx$.

Solution.

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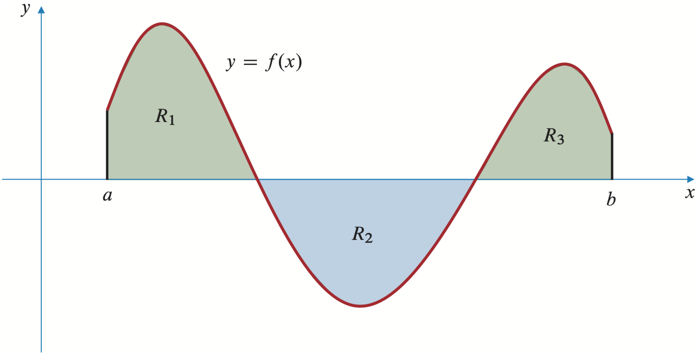
$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{(2n+1)(n+1)a^3}{6n^2} = \frac{a^3}{3}.$$

If $L(f, P_n) \leq I \leq U(f, P_n)$, we must have $I = \dfrac{a^3}{3}$. Thus, $f(x) = x^2$ is integrable over $[0, a]$, and

$$\int_0^a x^2 dx = \frac{a^3}{3}.$$

The Definite Integral



$$\int_a^b f(x)dx \text{ equals (area } R_1) - (\text{area } R_2) + (\text{area } R_3)$$

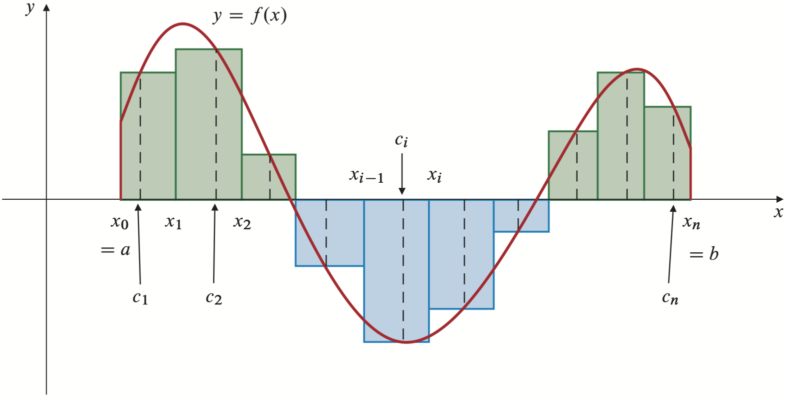
Let $P = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$ having norm $\|P\| = \max_{1 \le i \le n} \Delta x_i$. In each subinterval $[x_{i-1}, x_i]$ of P pick a point c_i (called a *tag*). Let $c = (c_1, c_2, \ldots, c_n)$ denote the set of these tags. The sum

$$\begin{aligned} R(f, P, c) &= \sum_{i=1}^{n} f(c_i)\Delta x_i \\ &= f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n \end{aligned}$$

is called the *Riemann sum* of f on $[a, b]$ corresponding to partition P and tags c.

The Definite Integral

General Riemann Sums



The Definite Integral

General Riemann Sums

$$L(f,P) \;\le\; R(f,P,c) \;\le\; U(f,P).$$

$$\Big\downarrow \quad \text{if } f \text{ is integrable on } [a,b]$$

$$\lim_{\substack{n(P)\to\infty \\ \|P\|\to 0}} R(f,P,c) = \int_a^b f(x)\,dx.$$

The Definite Integral

Theorem

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

The Definite Integral

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$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(1 + \frac{2i-1}{n}\right)^{1/3} = \int_0^2 (1+x)^{1/3}dx.$$