MAT123 MATHEMATICS I

Lecture 17: Integration (Continued)

Outline

Integration

Properties of the Definite Integral

Let f and g be integrable on an interval containing the points a, b, and c. Then

(a) An integral over an interval of zero length is zero:

$$\int_a^a f(x)dx = 0.$$

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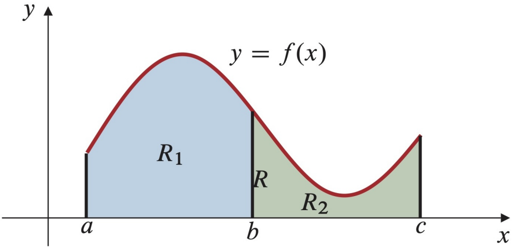
(c) An integral depends linearly on the integrand. If A and B are constants, then

$$\int_a^b (Af(x) + Bg(x))\,dx = A\int_a^b f(x)dx + B\int_a^b g(x)dx.$$

Properties of the Definite Integral

(d) An integral depends additively on the interval of integration.

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$$



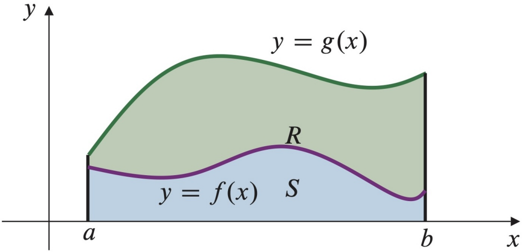
area R_1 + area R_2 = area R

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Properties of the Definite Integral

(e) If $a \le b$ and $f(x) \le g(x)$ for $a \le x \le b$, then

$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$



area $S \le$ area R

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

Properties of the Definite Integral

(f) The triangle inequality for sums extends to definite integrals. If $a \leq b$, then

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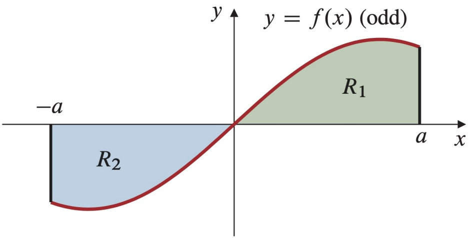
(g) The integral of an odd function over an interval symmetric about zero is zero. If f is an odd function (i.e., $f(-x) = -f(x)$ for all x), then $\int_{-a}^{a} f(x)dx = 0$.

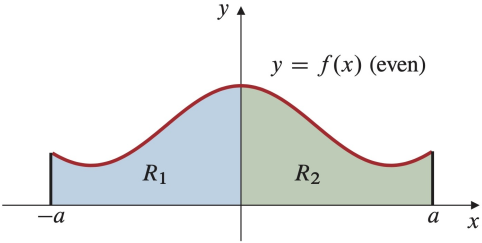


area R_1 − area $R_2 = 0$

$$\int_{-a}^{a} f(x)\,dx = 0$$

(h) The integral of an even function over an interval symmetric about zero is twice the integral over the positive half of the interval. If f is an even function (i.e., $f(-x) = f(x)$ for all x), then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$



area R_1 + area $R_2 = 2 \times$ area R_2

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Properties of the Definite Integral

Example. Evaluate

$$\text{(a)} \int_{-2}^{2}(2+5x)dx, \quad \text{(b)} \int_{0}^{3}(2+x)dx, \quad \text{(c)} \int_{-3}^{3}\sqrt{9-x^2}dx.$$

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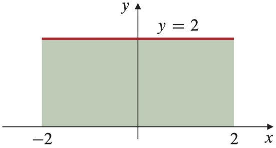

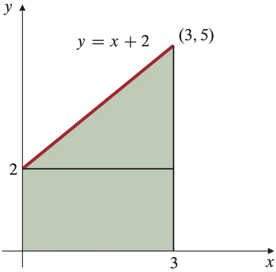

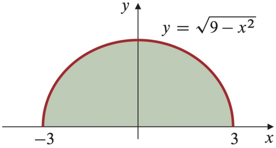

Properties of the Definite Integral

A Mean-Value Theorem for Integrals

Theorem. *Mean-Value Theorem for Integrals*

If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

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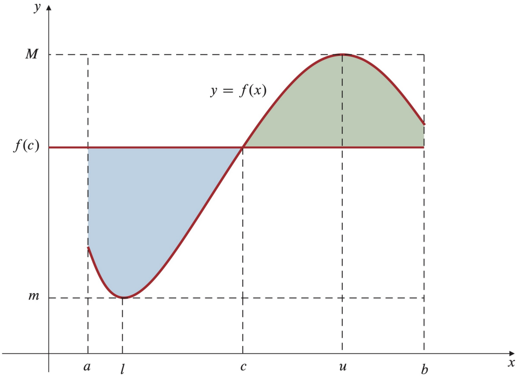
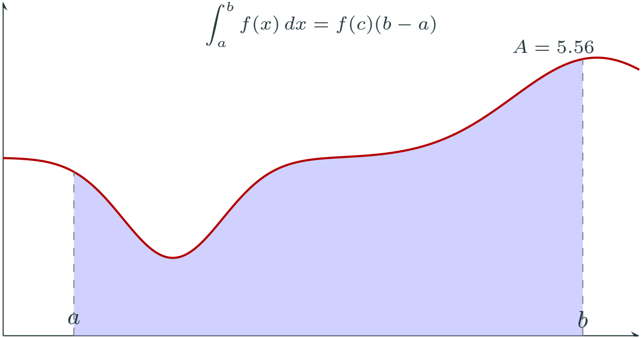



$\int_a^b f(x)\,dx = f(c)(b-a)$
$A = 5.56$
a
b

Properties of the Definite Integral

Definition. *Average value of a function*

If f is integrable on $[a, b]$, then the *average value* or *mean value* of f on $[a, b]$, denoted by $\bar{f}$, is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)dx.$$

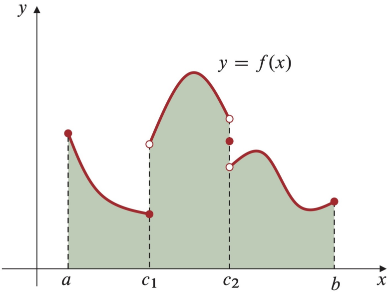
Example. Find the average value of the function $f(x) = 2x$ on the interval $[1, 5]$.

Properties of the Definite Integral

Piecewise continuous functions

Let $c_0 < c_1 < \cdots < c_{n-1} < c_n$ be a finite set of points on the real line. A function f defined on $[c_0, c_n]$ except possibly at some of the points c_i, $(0 \le i \le n)$, is called *piecewise continuous* on that interval if for each i $(1 \le i \le n)$ there exists a function F_i continuous on the *closed* interval $[c_{i-1}, c_i]$ such that

$$f(x) = F_i(x) \text{ on the } \textit{open} \text{ interval } (c_{i-1}, c_i).$$



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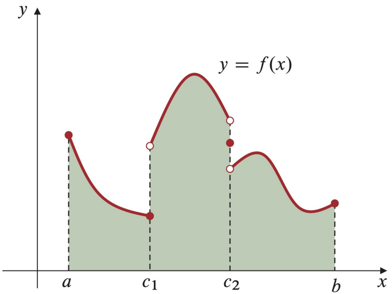
In this case

$$\int_{c_0}^{c_n} f(x)dx = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} F_i(x)dx$$



Properties of the Definite Integral

Example. Find $\int_0^3 f(x)dx$, where

$$f(x) = \begin{cases} \sqrt{1-x^2} & \text{if } 0 \leq x \leq 1, \\ 2 & \text{if } 1 < x \leq 2, \\ x-2 & \text{if } 2 < x \leq 3. \end{cases}$$

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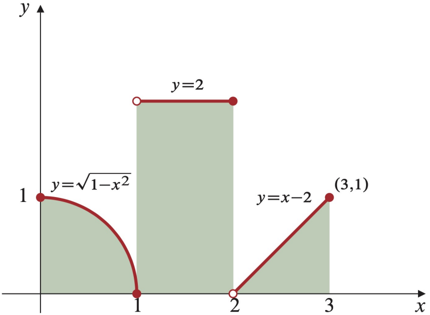

The Fundamental Theorem of Calculus

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Suppose that the function f is continuous on an interval I containing the point a.

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PART II. If $G(x)$ is any antiderivative of $f(x)$ on I, so that $G'(x) = f(x)$ on I, then for any b in I we have

$$\int_a^b f(t)dt = G(b) - G(a).$$

The Fundamental Theorem of Calculus

Definition

To facilitate the evaluation of definite integrals using the Fundamental Theorem of Calculus, we define the *evaluation symbol*:

$$F(x)\Big|_a^b = F(b) - F(a).$$

The Fundamental Theorem of Calculus

Example. Evaluate (a) $\int_0^a x^2 dx$ and (b) $\int_{-1}^2 (x^2 - 3x + 2) dx$.

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Solution.

$$\text{(a)}\ \int_0^a x^2 dx = \frac{1}{3}x^3\Big|_0^a = \frac{1}{3}a^3 - \frac{1}{3}0^3 = \frac{a^3}{3} \quad \text{(because } \frac{d}{dx}\frac{x^3}{3} = x^2\text{).}$$

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(b) $$\int_{-1}^2 (x^2 - 3x + 2)dx = \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right)\bigg|_{-1}^2$$
$$= \frac{1}{3}(8) - \frac{3}{2}(4) + 2(2) - \left(\frac{1}{3}(-1) - \frac{3}{2}(1) + 2(-1)\right) = \frac{9}{2}.$$

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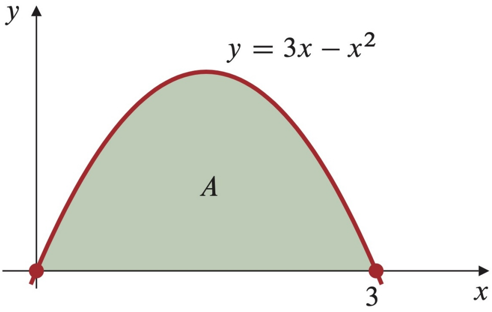

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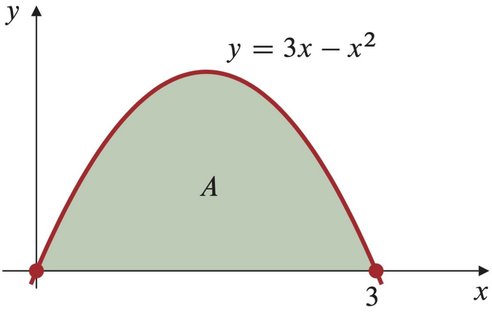

$$A = \int_0^3 (3x - x^2)dx = \left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^3 = \left(\frac{27}{2} - \frac{27}{3}\right) - (0 - 0) = \frac{9}{2}.$$

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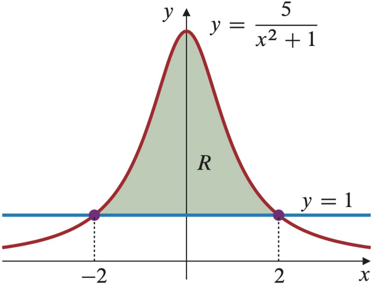

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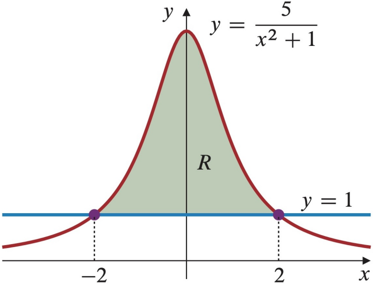

$$A = \int_{-2}^{2} \frac{5}{x^2+1}\,dx - 4 = 2\int_{0}^{2} \frac{5}{x^2+1}\,dx - 4$$

$$= 10\tan^{-1} x\Big|_0^2 - 4 = 10\tan^{-1} 2 - 4.$$

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even though $1/x$ is an odd function. In fact, $1/x$ is undefined and has no limit at $x = 0$, and it is not integrable on $[-1, 0]$ or $[0, 1]$.

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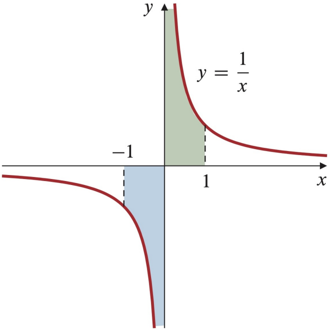

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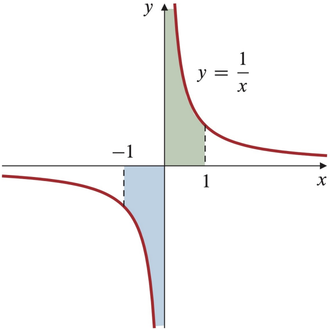
Observe that

$$\lim_{c \to 0^+} \int_c^1 \frac{1}{x}\,dx = \lim_{c \to 0^+} (-\ln c) = \infty,$$

and similarly $\displaystyle\lim_{d \to 0^-} \int_{-1}^{d} \frac{1}{x}\,dx = -\infty.$ So both shaded regions in the figure have infinite area.



The Fundamental Theorem of Calculus

Example. Find the derivatives of the following functions:

$$\text{(a) } F(x) = \int_x^3 e^{-t^2} dt, \quad \text{(b) } G(x) = x^2 \int_{-4}^{5x} e^{-t^2}, \quad \text{(c) } H(x) = \int_{x^2}^{x^3} e^{-t^2} dt.$$

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(b) By the Product Rule and the Chain Rule,

$$\begin{aligned} G'(x) &= 2x \int_{-4}^{5x} e^{-t^2} dt + x^2 \frac{d}{dx} \int_{-4}^{5x} e^{-t^2} dt \\ &= 2x \int_{-4}^{5x} e^{-t^2} dt + x^2 e^{-(5x)^2}(5) \\ &= 2x \int_{-4}^{5x} e^{-t^2} dt + 5x^2 e^{-25x^2}. \end{aligned}$$

The Fundamental Theorem of Calculus

(c) Split the integral into a difference of two integrals in each of which the variable x appears only in the upper limit.

$$H(x) = \int_0^{x^3} e^{-t^2}\,dt - \int_0^{x^2} e^{-t^2}\,dt.$$

Then

$$H'(x) = e^{-(x^3)^2}(3x^2) - e^{-(x^2)^2}(2x) = 3x^2 e^{-x^6} - 2xe^{-x^4}.$$

Example. Evaluate $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \cos\left(\frac{j\pi}{2n}\right)$.

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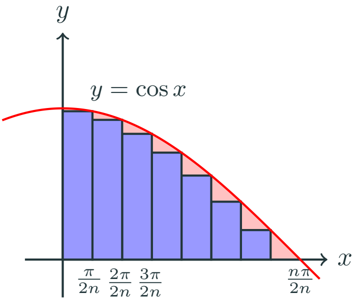
The sum involves values of $\cos x$ at the right endpoints of the n subintervals of the partition $0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \cdots, \frac{n\pi}{2n}$ of the interval $[0, \pi/2]$. Therefore $\Delta x = \frac{\pi}{2n}$.



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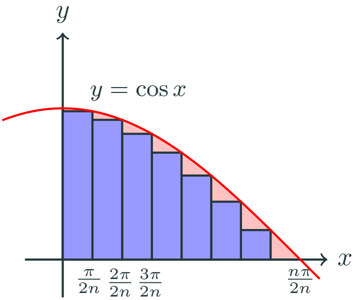
$$\lim_{n\to\infty} \frac{\pi}{2n} \sum_{j=1}^{n} \cos\left(\frac{j\pi}{2n}\right) = \int_0^{\pi/2} \cos x \, dx$$

$$= \sin x \Big|_0^{\pi/2} = 1 - 0 = 1.$$



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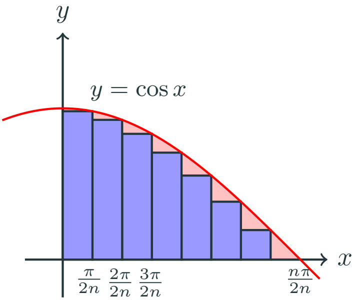

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Hence, $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \cos\left(\frac{j\pi}{2n}\right) = \frac{2}{\pi}$.

The Method of Substitution

Definition

The *indefinite integral* of $f(x)$ on interval I is

$$\int f(x)\,dx = F(x) + C \quad \text{on } I,$$

provided $F'(x) = f(x)$ for all $x \in I$.

The Method of Substitution

Some elementary integrals

1. $\int 1\,dx = x + C$
2. $\int x\,dx = \frac{1}{2}x^2 + C$
3. $\int x^2\,dx = \frac{1}{3}x^3 + C$
4. $\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$
5. $\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$
6. $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$
7. $\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C \quad (r \neq -1)$
8. $\int \frac{1}{x}\,dx = \ln|x| + C$
9. $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$
10. $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$
11. $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$
12. $\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$
13. $\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax + C$
14. $\int \csc ax\cot ax\,dx = -\frac{1}{a}\csc ax + C$
15. $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C \quad (a > 0)$
16. $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
17. $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
18. $\int b^{ax}\,dx = \frac{1}{a\ln b}b^{ax} + C$
19. $\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$
20. $\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$

Example. Combining elementary integrals

(a) $\int (x^4 - 3x^3 + 8x^2 - 6x - 7)dx = \frac{x^5}{5} - \frac{3x^4}{4} + \frac{8x^3}{3} - 3x^2 - 7x + C$

The Method of Substitution

Example. Combining elementary integrals

(a) $\int (x^4 - 3x^3 + 8x^2 - 6x - 7)dx = \frac{x^5}{5} - \frac{3x^4}{4} + \frac{8x^3}{3} - 3x^2 - 7x + C$

(b) $\int \left(5x^{3/5} - \frac{3}{2 + x^2}\right) dx = \frac{25}{8}x^{8/5} - \frac{3}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + C$

Example. Combining elementary integrals

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(c) $\displaystyle\int(4\cos 5x - 5\sin 3x)dx = \frac{4}{5}\sin 5x + \frac{5}{3}\cos 3x + C$

Example. Combining elementary integrals

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(c) $\displaystyle\int (4\cos 5x - 5\sin 3x)dx = \frac{4}{5}\sin 5x + \frac{5}{3}\cos 3x + C$

(d) $\displaystyle\int \left(\frac{1}{\pi x} + a^{\pi x}\right) dx = \frac{1}{\pi}\ln|x| + \frac{1}{\pi \ln a}a^{\pi x} + C,\ (a > 0).$

The Method of Substitution

Example. Find $\int \frac{(x+1)^3}{x} dx$.

Solution.

The Method of Substitution

Example. Find $\int \frac{(x+1)^3}{x} dx$.

Solution.

$$\int \frac{(x+1)^3}{x} dx = \int \frac{x^3 + 3x^2 + 3x + 1}{x} dx$$

The Method of Substitution

Example. Find $\int \frac{(x+1)^3}{x} dx$.

Solution.

$$\begin{aligned}\int \frac{(x+1)^3}{x} dx &= \int \frac{x^3+3x^2+3x+1}{x} dx \\ &= \int \left(x^2+3x+3+\frac{1}{x}\right) dx\end{aligned}$$

The Method of Substitution

Example. Find $\int \frac{(x+1)^3}{x} dx$.

Solution.

$$\begin{aligned}\int \frac{(x+1)^3}{x} dx &= \int \frac{x^3+3x^2+3x+1}{x} dx \\ &= \int \left(x^2+3x+3+\frac{1}{x}\right) dx \\ &= \frac{1}{3}x^3+\frac{3}{2}x^2+3x+\ln|x|+C.\end{aligned}$$