MAT123 MATHEMATICS I

Lecture 18: Integration (Continued)

Outline

The Method of Substitution

Trigonometric Integrals

Area of Plane Regions

The Method of Substitution

The Method of Substitution

$$\boxed{\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)}$$

The Method of Substitution

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$$

$$\downarrow$$

$$\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C$$

The Method of Substitution

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Let $u = g(x)$. Then $du/dx = g'(x)$, or in differential form, $du = g'(x)dx$.

The Method of Substitution

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$$\boxed{\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C}$$

Let $u = g(x)$. Then $du/dx = g'(x)$, or in differential form, $du = g'(x)dx$. Thus,

$$\int f'(\underbrace{g(x)}_{\color{red}{u}})\, \underbrace{g'(x)dx}_{\color{red}{du}} = \int f'(u)du = f(u) + C = f(g(x)) + C.$$

Example. Find the indefinite integrals:

$$\text{(a)} \int \frac{x}{x^2+1}\,dx, \quad \text{(b)} \int \frac{\sin(3\ln x)}{x}\,dx, \quad \text{(c)} \int e^x\sqrt{1+e^x}\,dx.$$

Solution.

The Method of Substitution

Example. Find the indefinite integrals:

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Solution. (a) Let $u = x^2 + 1$. Then $du = 2x\,dx$.

The Method of Substitution

Example. Find the indefinite integrals:

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Solution. (a) Let $u = x^2 + 1$. Then $du = 2x\,dx$. So

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{du}{u}$$

Example. Find the indefinite integrals:

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Solution. (a) Let $u = x^2 + 1$. Then $du = 2x\,dx$. So

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$$
$$= \ln\sqrt{x^2+1} + C.$$

The Method of Substitution

Example. Find the indefinite integrals:

$$\text{(a)} \int \frac{x}{x^2+1}\,dx, \quad \text{(b)} \int \frac{\sin(3\ln x)}{x}\,dx, \quad \text{(c)} \int e^x\sqrt{1+e^x}\,dx.$$

Solution. (b) Let $u = 3\ln x$. Then $du = \dfrac{3}{x}dx$.

The Method of Substitution

Example. Find the indefinite integrals:

$$\text{(a)} \int \frac{x}{x^2+1}\,dx, \quad \text{(b)} \int \frac{\sin(3\ln x)}{x}\,dx, \quad \text{(c)} \int e^x\sqrt{1+e^x}\,dx.$$

Solution. (b) Let $u = 3\ln x$. Then $du = \dfrac{3}{x}dx$. So

$$\int \frac{\sin(3\ln x)}{x}\,dx = \frac{1}{3}\int \sin u\,du$$

Example. Find the indefinite integrals:

$$\text{(a)} \int \frac{x}{x^2+1}\,dx, \quad \text{(b)} \int \frac{\sin(3\ln x)}{x}\,dx, \quad \text{(c)} \int e^x\sqrt{1+e^x}\,dx.$$

Solution. (b) Let $u = 3\ln x$. Then $du = \frac{3}{x}dx$. So

$$\int \frac{\sin(3\ln x)}{x}\,dx = \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(3\ln x) + C.$$

Example. Find the indefinite integrals:

$$\text{(a)} \int \frac{x}{x^2+1}\,dx, \quad \text{(b)} \int \frac{\sin(3\ln x)}{x}\,dx, \quad \text{(c)} \int e^x\sqrt{1+e^x}\,dx.$$

Solution. (c) Let $v = 1 + e^x$. Then $dv = e^x\,dx$.

Example. Find the indefinite integrals:

$$\text{(a)} \int \frac{x}{x^2+1}dx, \quad \text{(b)} \int \frac{\sin(3\ln x)}{x}dx, \quad \text{(c)} \int e^x\sqrt{1+e^x}dx.$$

Solution. (c) Let $v = 1 + e^x$. Then $dv = e^x dx$. So

$$\int e^x\sqrt{1+e^x}dx = \int v^{1/2}dv$$

The Method of Substitution

Example. Find the indefinite integrals:

$$\text{(a)} \int \frac{x}{x^2+1}dx, \quad \text{(b)} \int \frac{\sin(3\ln x)}{x}dx, \quad \text{(c)} \int e^x\sqrt{1+e^x}dx.$$

Solution. (c) Let $v = 1 + e^x$. Then $dv = e^x dx$. So

$$\int e^x\sqrt{1+e^x}dx = \int v^{1/2}dv = \frac{2}{3}v^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C.$$

Example. Find the indefinite integrals:

$$\text{(a)} \int \frac{1}{x^2+4x+5}\,dx, \quad \text{(b)} \int \frac{dx}{\sqrt{e^{2x}-1}}.$$

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Example. Find the indefinite integrals:

$$\text{(a)} \int \frac{1}{x^2+4x+5}\,dx, \quad \text{(b)} \int \frac{dx}{\sqrt{e^{2x}-1}}.$$

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Let $t = x + 2$ Then $dt = dx$

The Method of Substitution

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Solution.

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Let $t = x+2$ Then $dt = dx$

$$= \int \frac{dt}{t^2+1}$$

Example. Find the indefinite integrals:

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Let $t = x+2$ Then $dt = dx$

$$= \int \frac{dt}{t^2+1}$$

$$= \tan^{-1} t + C = \tan^{-1}(x+2) + C.$$

Example. Find the indefinite integrals:

$$\text{(a)} \int \frac{1}{x^2+4x+5}\,dx, \quad \text{(b)} \int \frac{dx}{\sqrt{e^{2x}-1}}.$$

Solution.

$$\text{(b)} \int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$$

Example. Find the indefinite integrals:

$$\text{(a)} \int \frac{1}{x^2+4x+5}\,dx, \quad \text{(b)} \int \frac{dx}{\sqrt{e^{2x}-1}}.$$

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$$\text{(b)} \int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$$
$$= \int \frac{e^{-x}\,dx}{\sqrt{1-(e^{-x})^2}}$$

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The Method of Substitution

Example. Find the indefinite integrals:

$$\text{(a)} \int \frac{1}{x^2 + 4x + 5}\, dx, \quad \text{(b)} \int \frac{dx}{\sqrt{e^{2x} - 1}}.$$

Solution.

$$\text{(b)} \int \frac{dx}{\sqrt{e^{2x} - 1}} = \int \frac{dx}{e^x \sqrt{1 - e^{-2x}}}$$

$$= \int \frac{e^{-x}\, dx}{\sqrt{1 - (e^{-x})^2}}$$

Let $u = e^{-x}$ Then $du = -e^{-x} dx$

$$= -\int \frac{du}{\sqrt{1 - u^2}}$$

Example. Find the indefinite integrals:

$$\text{(a)} \int \frac{1}{x^2+4x+5}\,dx, \quad \text{(b)} \int \frac{dx}{\sqrt{e^{2x}-1}}.$$

Solution.

$$\begin{aligned}\text{(b)} \int \frac{dx}{\sqrt{e^{2x}-1}} &= \int \frac{dx}{e^x\sqrt{1-e^{-2x}}} \\ &= \int \frac{e^{-x}\,dx}{\sqrt{1-(e^{-x})^2}} \\ &= -\int \frac{du}{\sqrt{1-u^2}} \\ &= -\sin^{-1} u + C = -\sin^{-1}(e^{-x}) + C.\end{aligned}$$

Let $u = e^{-x}$ Then $du = -e^{-x}dx$

The Method of Substitution

Theorem. *Substitution in a definite integral*

Suppose that g is a differentiable function on $[a, b]$ that satisfies $g(a) = A$ and $g(b) = B$. Also suppose that f is continuous on the range of g. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_A^B f(u)\,du.$$

The Method of Substitution

Example. Evaluate the integral $I = \displaystyle\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx$.

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Solution. METHOD I. Let $u = \sqrt{x+1}$. Then $du = \dfrac{dx}{2\sqrt{x+1}}$. If $x = 0$, then $u = 1$; if $x = 8$, then $u = 3$.

Example. Evaluate the integral $I = \int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx$.

Solution. METHOD I. Let $u = \sqrt{x+1}$. Then $du = \dfrac{dx}{2\sqrt{x+1}}$. If $x = 0$, then $u = 1$; if $x = 8$, then $u = 3$. Thus

$$I = 2\int_1^3 \cos u\,du = 2\sin u\Big|_1^3 = 2\sin 3 - 2\sin 1.$$

The Method of Substitution

Example. Evaluate the integral $I = \int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx$.

Solution. **METHOD II.** We use the same substitution as in Method I, but we do not transform the limits of integration from x values to u values. Hence, we must return to the variable x before substituting in the limits:

$$I = 2\int_{x=0}^{x=8} \cos u\,du = 2\sin u\Big|_{x=0}^{x=8} = 2\sin\sqrt{x+1}\Big|_0^8 = 2\sin 3 - 2\sin 1.$$

The Method of Substitution

Example. Find the area of the region bounded by $y = \left(2 + \sin \frac{x}{2}\right)^2 \cos \frac{x}{2}$, the x-axis, and the lines $x = 0$ and $x = \pi$.

Solution.

The Method of Substitution

Example. Find the area of the region bounded by $y = \left(2 + \sin \frac{x}{2}\right)^2 \cos \frac{x}{2}$, the x-axis, and the lines $x = 0$ and $x = \pi$.

Solution. Since $y \geq 0$ when $0 \leq x \leq \pi$, the required area is

$$A = \int_0^{\pi} \left(2 + \sin \frac{x}{2}\right)^2 \cos \frac{x}{2} dx$$

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Let $v = 2 + \sin\frac{x}{2}$. Then $dv = \frac{1}{2}\cos\frac{x}{2}\, dx$

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$$= 2\int_2^3 v^2\,dv = \frac{2}{3}v^3\Big|_2^3 = \frac{2}{3}(27 - 8) = \frac{38}{3} \text{ square units.}$$

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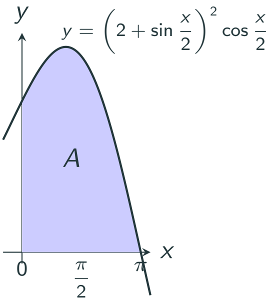

Integrals of tangent, cotangent, secant, and cosecant

$$\int \tan x \, dx = \ln|\sec x| + C,$$

$$\int \cot x \, dx = \ln|\sin x| + C = -\ln|\csc x| + C,$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C,$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C = \ln|\csc x - \cot x| + C.$$

The Method of Substitution

Trigonometric Integrals

- $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

- $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$ Let $u = \cos x$. Then $du = -\sin x \, dx$

- $$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx \quad \text{Let } u = \cos x. \text{ Then } du = -\sin x \, dx$$
  $$= -\int \frac{du}{u} = -\ln|u| + C$$
  $$= -\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C.$$

The integral of $\sec x$ can be evaluated by rewriting it in the form

$$\int \sec x\, dx = \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}\, dx$$

and using the substitution $u = \sec x + \tan x \Rightarrow du = (\sec x(\sec x + \tan x))\, dx$.
The integral of $\csc x$ can be evaluated similarly.

The Method of Substitution

Trigonometric Integrals

We now consider integrals of the form

$$\int \sin^m x \cos^n x \, dx.$$

Case 1. If either m or n is positive odd integer, the integral can be done easily by substitution.

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Solution. (a) $\displaystyle\int \sin^3 x \cos^8 x \, dx$

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Example. Evaluate (a) $\int \sin^3 x \cos^8 x \, dx$ and (b) $\int \cos^5 ax \, dx$.

Solution. (a) $\int \sin^3 x \cos^8 x \, dx = \int (1 - \cos^2 x) \cos^8 x \sin x \, dx$

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$u = \cos x \Rightarrow du = -\sin x \, dx$

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Example. Evaluate (a) $\int \sin^3 x \cos^8 x \, dx$ and (b) $\int \cos^5 ax \, dx$.

Solution.

$$\text{(a)} \int \sin^3 x \cos^8 x \, dx = \int (1 - \cos^2 x) \cos^8 x \sin x \, dx$$

$u = \cos x \Rightarrow du = -\sin x \, dx$

$$= -\int (1 - u^2) u^8 \, du = \int (u^{10} - u^8) \, du$$

$$= \frac{u^{11}}{11} - \frac{u^9}{9} + C = \frac{\cos^{11} x}{11} - \frac{\cos^9 x}{9} + C.$$

We now consider integrals of the form

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Case 1. If either m or n is positive odd integer, the integral can be done easily by substitution.

Example. Evaluate (a) $\int \sin^3 x \cos^8 x \, dx$ and (b) $\int \cos^5 ax \, dx$.

Solution.

$$\text{(b)} \int \cos^5 ax \, dx = \int (1 - \sin^2 ax)^2 \cos ax \, dx$$

We now consider integrals of the form

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Case 1. If either m or n is positive odd integer, the integral can be done easily by substitution.

Example. Evaluate (a) $\int \sin^3 x \cos^8 x \, dx$ and (b) $\int \cos^5 ax \, dx$.

Solution. (b) $\int \cos^5 ax \, dx = \int (1 - \sin^2 ax)^2 \cos ax \, dx$

$u = \sin ax \Rightarrow du = a \cos ax \, dx$

We now consider integrals of the form

$$\int \sin^m x \cos^n x \, dx.$$

Case 1. If either m or n is positive odd integer, the integral can be done easily by substitution.

Example. Evaluate (a) $\int \sin^3 x \cos^8 x \, dx$ and (b) $\int \cos^5 ax \, dx$.

Solution.

$$\text{(b)} \int \cos^5 ax \, dx = \int (1 - \sin^2 ax)^2 \cos ax \, dx$$

$$u = \sin ax \Rightarrow du = a \cos ax \, dx \quad = \frac{1}{a} \int (1 - u^2)^2 \, du = \frac{1}{a} \int (1 - 2u^2 + u^4) \, du$$

We now consider integrals of the form

$$\int \sin^m x \cos^n x \, dx.$$

Case 1. If either m or n is positive odd integer, the integral can be done easily by substitution.

Example. Evaluate (a) $\int \sin^3 x \cos^8 x \, dx$ and (b) $\int \cos^5 ax \, dx$.

Solution.

$$\text{(b)} \int \cos^5 ax \, dx = \int (1 - \sin^2 ax)^2 \cos ax \, dx$$

$u = \sin ax \Rightarrow du = a \cos ax \, dx$

$$= \frac{1}{a} \int (1 - u^2)^2 \, du = \frac{1}{a} \int (1 - 2u^2 + u^4) \, du$$

$$= \frac{1}{a} \left(u - \frac{2u^3}{3} + \frac{u^5}{5} \right) + C$$

$$= \frac{1}{a} \left(\sin ax - \frac{2 \sin^3 ax}{3} + \frac{\sin^5 ax}{5} \right) + C.$$

Case 2. If the powers of $\sin x$ and $\cos x$ are both even, we can make use of the double-angle formulas

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad \text{and} \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

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$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad \text{and} \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

Example. Evaluate $\int \sin^4 x \, dx$.

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$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad \text{and} \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

Example. Evaluate $\displaystyle\int \sin^4 x \, dx$.

Solution.

$$\int \sin^4 x \, dx = \frac{1}{4} \int (1 - \cos 2x)^2 \, dx$$

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Example. Evaluate $\displaystyle\int \sin^4 x \, dx$.

Solution.

$$\int \sin^4 x \, dx = \frac{1}{4} \int (1 - \cos 2x)^2 \; dx = \frac{1}{4} \int \left(1 - 2\cos 2x + \cos^2 2x\right) \; dx$$

Case 2. If the powers of $\sin x$ and $\cos x$ are both even, we can make use of the double-angle formulas

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) \quad \text{and} \quad \sin^2 x = \frac{1}{2}(1-\cos 2x).$$

Example. Evaluate $\int \sin^4 x\, dx$.

Solution.

$$\int \sin^4 x\, dx = \frac{1}{4}\int (1-\cos 2x)^2\, dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right)\, dx$$
$$= \frac{1}{4}\left(x - \sin 2x + \frac{1}{2}\int (1+\cos 4x)\, dx\right)$$

Case 2. If the powers of $\sin x$ and $\cos x$ are both even, we can make use of the double-angle formulas

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad \text{and} \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

Example. Evaluate $\int \sin^4 x \, dx$.

Solution.

$$\begin{aligned}
\int \sin^4 x \, dx &= \frac{1}{4} \int (1 - \cos 2x)^2 \, dx = \frac{1}{4} \int \left(1 - 2\cos 2x + \cos^2 2x\right) \, dx \\
&= \frac{1}{4} \left(x - \sin 2x + \frac{1}{2} \int (1 + \cos 4x) \, dx\right) \\
&= \frac{1}{4} \left(x - \sin 2x + \frac{1}{2} \left(x + \frac{1}{4} \sin 4x\right)\right) + C \\
&= \frac{1}{8}x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + \frac{1}{8} + C.
\end{aligned}$$

$\int \sec^m x \tan^n x \, dx$ or $\int \csc^m x \cot^n x \, dx$, unless m is odd and n is even.

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Example. Evaluate the following integrals:

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Solution.

(a) $\int \tan^2 x \, dx = \int (\sec^2 x - 1) dx = \tan x - x + C$.

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Solution.

(a) $\int \tan^2 x \, dx = \int (\sec^2 x - 1) dx = \tan x - x + C.$

(b) $\int \sec^4 t \, dt = \int (1 + \tan^2 t) \sec^2 t \, dt$

$\int \sec^m x \tan^n x \, dx$ or $\int \csc^m x \cot^n x \, dx$, unless m is odd and n is even.

Example. Evaluate the following integrals:

(a) $\int \tan^2 x \, dx$ (b) $\int \sec^4 t \, dt$ (c) $\int \sec^3 x \tan^3 x \, dx$.

Solution.

(a) $\int \tan^2 x \, dx = \int (\sec^2 x - 1) dx = \tan x - x + C.$

(b) $\int \sec^4 t \, dt = \int (1 + \tan^2 t) \sec^2 t \, dt$ Let $u = \tan t$. Then $du = \sec^2 t \, dt$

$\int \sec^m x \tan^n x \, dx$ or $\int \csc^m x \cot^n x \, dx$, unless m is odd and n is even.

Example. Evaluate the following integrals:

(a) $\int \tan^2 x \, dx$ (b) $\int \sec^4 t \, dt$ (c) $\int \sec^3 x \tan^3 x \, dx$.

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$$= \int (1 + u^2) du = u + \frac{u^3}{3} + C$$

$$= \tan t + \frac{\tan^3 t}{3} + C,$$

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(c)
$$\int \sec^3 x \tan^3 x \, dx$$
$$= \int \sec^2 x (\sec^2 x - 1) \sec x \tan x \, dx$$

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$$= \int \sec^2 x(\sec^2 x - 1) \sec x \tan x \, dx$$

Let $u = \sec x$. Then $du = \sec x \tan x \, dx$

$$= \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

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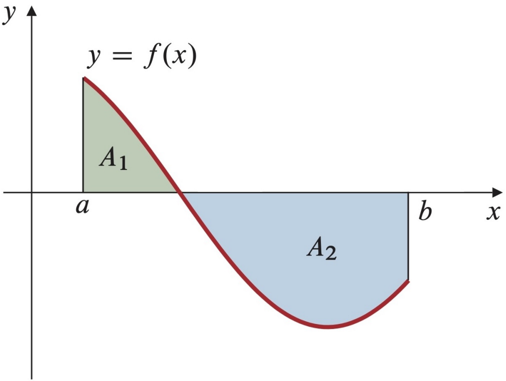
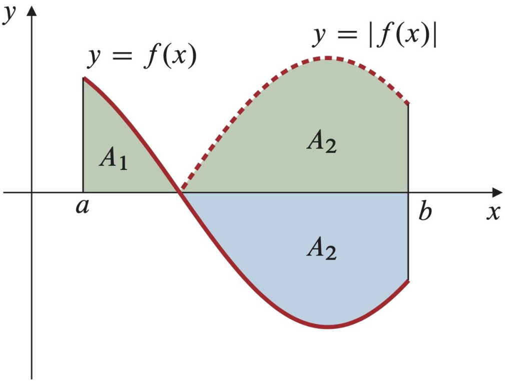
Area of Plane Regions

Areas of Plane Regions



$$\int_a^b f(x)dx = A_1 - A_2$$

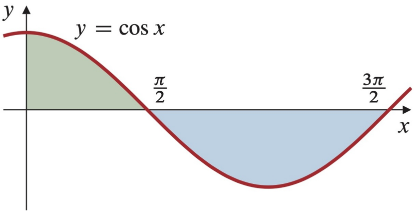
Areas of Plane Regions



$$\int_a^b f(x)dx = A_1 - A_2$$

$$\int_a^b |f(x)|dx = A_1 + A_2$$

Example. The area bounded by $y = \cos x$, $y = 0$, $x = 0$, and $x = 3\pi/2$ is



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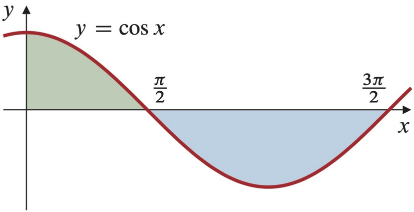
$$A = \int_0^{3\pi/2} |\cos x|\, dx$$



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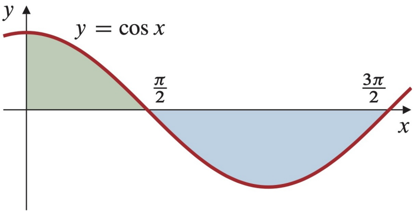
$$A = \int_0^{3\pi/2} |\cos x|\, dx$$

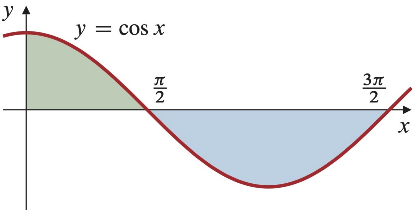
$$= \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{3\pi/2} (-\cos x)\, dx$$

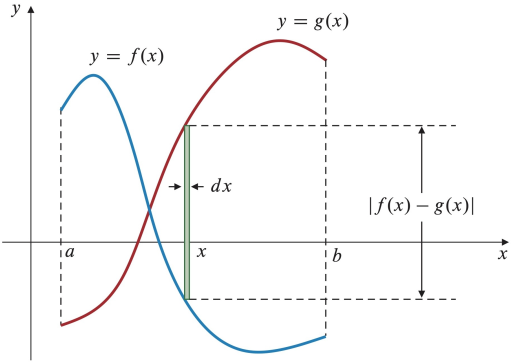


Areas of Plane Regions

Example. The area bounded by $y = \cos x$, $y = 0$, $x = 0$, and $x = 3\pi/2$ is

$$
\begin{aligned}
A &= \int_0^{3\pi/2} |\cos x|\, dx \\
&= \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{3\pi/2} (-\cos x)\, dx \\
&= \sin x\Big|_0^{\pi/2} - \sin x\Big|_{\pi/2}^{3\pi/2} \\
&= (1-0) - (-1-1) = 3 \text{ square units.}
\end{aligned}
$$





$$A = \int_a^b |f(x) - g(x)|\, dx$$

Areas of Plane Regions

Example. Find the area between the curves $y = x^2 - 2x$ and $y = 4 - x^2$.

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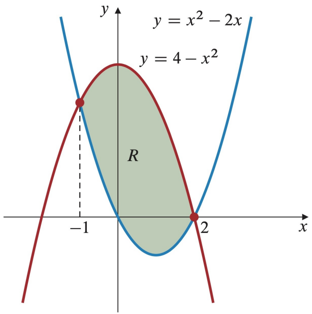

To find the points of intersection:

- $x^2 - 2x = y = 4 - x^2$

  $\Rightarrow 2x^2 - 2x - 4 = 0$

  $\Rightarrow 2(x - 2)(x + 1) = 0$

  $\Rightarrow x = 2$ or $x = -1$.

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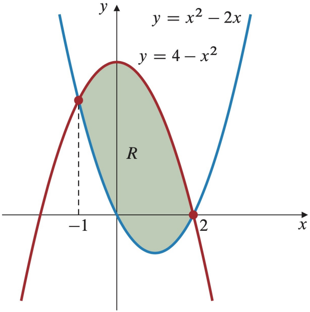

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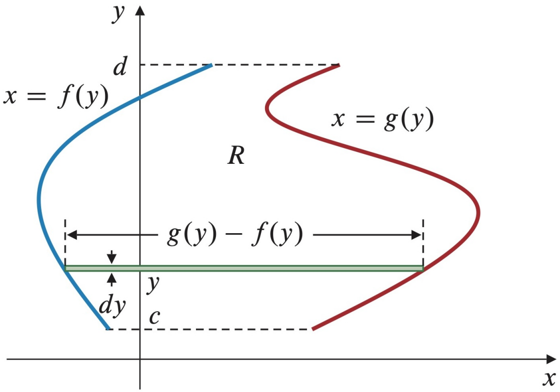
So the shaded area is

$$\begin{aligned} A &= \int_{-1}^{2} \left((4 - x^2) - (x^2 - 2x)\right) dx \\ &= \int_{-1}^{2} (4 - 2x^2 - 2x) dx \\ &= \left(4x - \frac{2}{3}x^3 + x^2\right)\Big|_{-1}^{2} \\ &= 4(2) - \frac{2}{3}(8) + 4 \\ &\quad - \left(-4 + \frac{2}{3} + 1\right) \\ &= 9. \end{aligned}$$

Areas of Plane Regions



$$A = \int_c^d (g(y) - f(y))dy.$$

Example. Find the area of the plane region lying to the right of the parabola $x = y^2 - 12$ and to the left of the straight line $y = x$.

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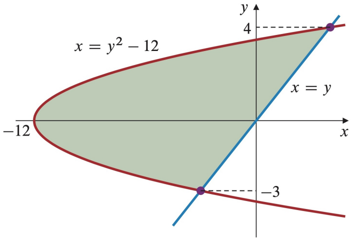

To find the points of intersection:

- $y^2 - 12 = x = y$

  $\Rightarrow y^2 - y - 12 = 0$

  $\Rightarrow (y - 4)(y + 3) = 0$

  $\Rightarrow y = 4$ or $y = -3$.

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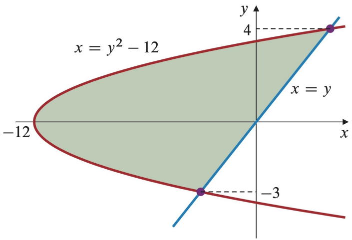

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- $y^2 - 12 = x = y$

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  $\Rightarrow y = 4$ or $y = -3$.

So the shaded area is

$$A = \int_{-3}^{4} \left(y - (y^2 - 12) \right) dy$$

$$= \left(\frac{y^2}{2} - \frac{y^3}{3} + 12y \right) \Big|_{-3}^{4}$$

$$= \frac{343}{6} \text{ square units.}$$

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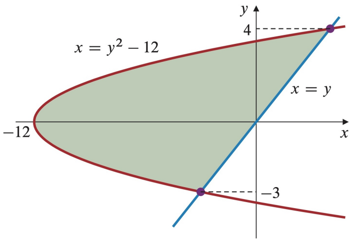

To find the points of intersection:

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$$A = \int_{-3}^{4} \Big(y - (y^2 - 12)\Big)\, dy$$

$$= \left(\frac{y^2}{2} - \frac{y^3}{3} + 12y\right)\Bigg|_{-3}^{4}$$

$$= \frac{343}{6} \text{ square units.}$$

We can find the same area by integrating in the x direction:

$$A = \int_{-12}^{-3} \Big(\sqrt{12+x} - (-\sqrt{12+x})\Big)\, dx + \int_{-3}^{4} (\sqrt{12+x} - x) dx.$$