MAT123 MATHEMATICS I

Lecture 19: Integration (Techniques of Integration)

Outline

Integration by Parts

Integrals of Rational Functions

- Linear and Quadratic Denominators
- Completing the Square
- Denominators with Repeated Factors

Integration by Parts

Integration by Parts

Suppose that $U(x)$ and $V(x)$ are two differentiable functions. According to the Product Rule,

$$\frac{d}{dx}\left(U(x)V(x)\right) = U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}.$$

Integration by Parts

Suppose that $U(x)$ and $V(x)$ are two differentiable functions. According to the Product Rule,

$$\frac{d}{dx}\left(U(x)V(x)\right) = U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}.$$

Integrating both sides of this equation and transposing terms, we obtain

$$\int U(x)\frac{dV}{dx}dx = U(x)V(x) - \int V(x)\frac{dU}{dx}dx,$$

or, more simply,

$$\int U\,dV = UV - \int V\,dU$$

Integration by Parts

Example. Evaluate the integral $I = \int xe^x \, dx$.

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Let $U = x$, $dV = e^x dx$.
Then $dU = dx$, $V = e^x$.

Integration by Parts

Example. Evaluate the integral $I = \int xe^x dx$.

Solution. We shall apply integration by parts.

$$I = \int xe^x dx = UV - \int V du$$
$$= xe^x - \int e^x dx$$
$$= xe^x - e^x + C = (x-1)e^x + C.$$

Let $U = x$, $dV = e^x dx$.
Then $dU = dx$, $V = e^x$.

Integration by Parts

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x \, dx, \quad \text{(b)} \int x^2 \sin x \, dx, \quad \text{(c)} \int x \tan^{-1} x \, dx, \quad \text{(d)} \int \sin^{-1} x \, dx$$

Integration by Parts

Example. Use integration by parts to evaluate:

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Solution.

$$\text{(a)} \quad \int \ln x \, dx$$

Integration by Parts

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x\, dx, \quad \text{(b)} \int x^2 \sin x\, dx, \quad \text{(c)} \int x \tan^{-1} x\, dx, \quad \text{(d)} \int \sin^{-1} x\, dx$$

Solution.

$$\text{(a)} \int \ln x\, dx$$

Let $U = \ln x$, $dV = dx$.
Then $dU = (1/x)dx$, $V = x$.

Integration by Parts

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x\, dx, \quad \text{(b)} \int x^2 \sin x\, dx, \quad \text{(c)} \int x \tan^{-1} x\, dx, \quad \text{(d)} \int \sin^{-1} x\, dx$$

Solution.

$$\text{(a)} \int \ln x\, dx$$

$$= x \ln x - \int x \frac{1}{x} dx$$

$$= x \ln x - x + C.$$

Let $U = \ln x$, $dV = dx$.
Then $dU = (1/x)dx$, $V = x$.

Integration by Parts

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x \, dx, \quad \text{(b)} \int x^2 \sin x \, dx, \quad \text{(c)} \int x \tan^{-1} x \, dx, \quad \text{(d)} \int \sin^{-1} x \, dx$$

Solution.

(b) $\int x^2 \sin x \, dx$ ←

Let $U = x^2$, $dV = \sin x \, dx$.
$\Rightarrow dU = 2x \, dx$, $V = -\cos x$.

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x \, dx, \quad \text{(b)} \int x^2 \sin x \, dx, \quad \text{(c)} \int x \tan^{-1} x \, dx, \quad \text{(d)} \int \sin^{-1} x \, dx$$

Solution.

$$\text{(b)} \int x^2 \sin x \, dx$$

Let $U = x^2, \; dV = \sin x \, dx.$
$\Rightarrow dU = 2x \, dx, \; V = -\cos x.$

$$= -x^2 \cos x + 2 \int x \cos x \, dx$$

Integration by Parts

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x\, dx, \quad \text{(b)} \int x^2 \sin x\, dx, \quad \text{(c)} \int x \tan^{-1} x\, dx, \quad \text{(d)} \int \sin^{-1} x\, dx$$

Solution.

(b) $\displaystyle\int x^2 \sin x\, dx$

Let $U = x^2,\ dV = \sin x\, dx.$
$\Rightarrow dU = 2x\, dx,\ V = -\cos x.$

$$= -x^2 \cos x + 2\int x \cos x\, dx$$

Let $U = x,\ dV = \cos x\, dx.$
$\Rightarrow dU = dx,\ V = \sin x.$

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x \, dx, \quad \text{(b)} \int x^2 \sin x \, dx, \quad \text{(c)} \int x \tan^{-1} x \, dx, \quad \text{(d)} \int \sin^{-1} x \, dx$$

Solution.

$$\text{(b)} \int x^2 \sin x \, dx$$

Let $U = x^2,\ dV = \sin x \, dx.$
$\Rightarrow dU = 2x \, dx,\ V = -\cos x.$

$$= -x^2 \cos x + 2 \int x \cos x \, dx$$

Let $U = x,\ dV = \cos x \, dx.$
$\Rightarrow dU = dx,\ V = \sin x.$

$$= -x^2 \cos x + 2 \left(x \sin x - \int \sin x \, dx \right)$$

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x\,dx, \quad \text{(b)} \int x^2 \sin x\,dx, \quad \text{(c)} \int x \tan^{-1} x\,dx, \quad \text{(d)} \int \sin^{-1} x\,dx$$

Solution.

$$\text{(b)} \int x^2 \sin x\,dx$$

Let $U = x^2,\ dV = \sin x\,dx.$
$\Rightarrow dU = 2x\,dx,\ V = -\cos x.$

$$= -x^2 \cos x + 2\int x \cos x\,dx$$

Let $U = x,\ dV = \cos x\,dx.$
$\Rightarrow dU = dx,\ V = \sin x.$

$$= -x^2 \cos x + 2\left(x \sin x - \int \sin x\,dx\right)$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + C.$$

Integration by Parts

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x\,dx, \quad \text{(b)} \int x^2 \sin x\,dx, \quad \text{(c)} \int x \tan^{-1} x\,dx, \quad \text{(d)} \int \sin^{-1} x\,dx$$

Solution.

(c) $\int x \tan^{-1} x\,dx$

Let $U = \tan^{-1} x,\ dV = x\,dx.$

$\Rightarrow dU = \dfrac{1}{1+x^2}\,dx,\ V = \dfrac{1}{2}x^2.$

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x\,dx, \quad \text{(b)} \int x^2 \sin x\,dx, \quad \text{(c)} \int x \tan^{-1} x\,dx, \quad \text{(d)} \int \sin^{-1} x\,dx$$

Solution.

(c) $\int x \tan^{-1} x\,dx$

Let $U = \tan^{-1} x,\ dV = x\,dx.$
$\Rightarrow dU = \dfrac{1}{1+x^2}\,dx,\ V = \dfrac{1}{2}x^2.$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

Integration by Parts

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x\, dx, \quad \text{(b)} \int x^2 \sin x\, dx, \quad \text{(c)} \int x \tan^{-1} x\, dx, \quad \text{(d)} \int \sin^{-1} x\, dx$$

Solution.

$$\text{(c)} \int x \tan^{-1} x\, dx$$

Let $U = \tan^{-1} x,\ dV = x\, dx.$
$\Rightarrow dU = \dfrac{1}{1+x^2}\, dx,\ V = \dfrac{1}{2}x^2.$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\, dx$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$$

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x\,dx, \quad \text{(b)} \int x^2 \sin x\,dx, \quad \text{(c)} \int x \tan^{-1} x\,dx, \quad \text{(d)} \int \sin^{-1} x\,dx$$

Solution.

(c) $\int x \tan^{-1} x\,dx$ ← Let $U = \tan^{-1} x$, $dV = x\,dx$. $\Rightarrow dU = \frac{1}{1+x^2}\,dx$, $V = \frac{1}{2}x^2$.

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x - \frac{1}{2}\tan^{-1} x + C$$

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x\, dx, \quad \text{(b)} \int x^2 \sin x\, dx, \quad \text{(c)} \int x \tan^{-1} x\, dx, \quad \text{(d)} \int \sin^{-1} x\, dx$$

Solution.

$$\text{(d)} \int \sin^{-1} x\, dx$$

Let $U = \sin^{-1} x,\ dV = x\, dx.$

$\Rightarrow dU = \dfrac{1}{\sqrt{1 - x^2}}\, dx,\ V = x.$

Integration by Parts

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x\, dx, \quad \text{(b)} \int x^2 \sin x\, dx, \quad \text{(c)} \int x \tan^{-1} x\, dx, \quad \text{(d)} \int \sin^{-1} x\, dx$$

Solution.

$$\text{(d)} \int \sin^{-1} x\, dx$$

Let $U = \sin^{-1} x,\ dV = x\, dx.$
$\Rightarrow dU = \dfrac{1}{\sqrt{1-x^2}}\, dx,\ V = x.$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Integration by Parts

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x \, dx, \quad \text{(b)} \int x^2 \sin x \, dx, \quad \text{(c)} \int x \tan^{-1} x \, dx, \quad \text{(d)} \int \sin^{-1} x \, dx$$

Solution.

$$\text{(d)} \int \sin^{-1} x \, dx$$

Let $U = \sin^{-1} x$, $dV = x\,dx$.
$\Rightarrow dU = \dfrac{1}{\sqrt{1-x^2}}\,dx$, $V = x$.

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Let $u = 1 - x^2$. Then $du = -2x\,dx$

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x\, dx, \quad \text{(b)} \int x^2 \sin x\, dx, \quad \text{(c)} \int x \tan^{-1} x\, dx, \quad \text{(d)} \int \sin^{-1} x\, dx$$

Solution.

$$\text{(d)} \int \sin^{-1} x\, dx$$

Let $U = \sin^{-1} x,\ dV = x\, dx.$
$\Rightarrow dU = \frac{1}{\sqrt{1-x^2}}\, dx,\ V = x.$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Let $u = 1 - x^2$. Then $du = -2x\, dx$

$$= x \sin^{-1} x + \frac{1}{2} \int u^{-1/2} du$$

Example. Use integration by parts to evaluate:

$$\text{(a)} \int \ln x \, dx, \quad \text{(b)} \int x^2 \sin x \, dx, \quad \text{(c)} \int x \tan^{-1} x \, dx, \quad \text{(d)} \int \sin^{-1} x \, dx$$

Solution.

$$\text{(d)} \int \sin^{-1} x \, dx$$

Let $U = \sin^{-1} x$, $dV = x\, dx$.
$\Rightarrow dU = \dfrac{1}{\sqrt{1-x^2}} dx$, $V = x$.

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Let $u = 1 - x^2$. Then $du = -2x \, dx$

$$= x \sin^{-1} x + \frac{1}{2} \int u^{-1/2} du$$

$$= x \sin^{-1} x + u^{1/2} + c$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + C$$

Integration by Parts

Example. Evaluate the integral $I = \int \sec^3 x \, dx$.

Solution. Start using integration by parts:

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$$I = \int \sec^3 x \, dx = \int \sec x \, \sec^2 x \, dx$$

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$$I = \int \sec^3 x \, dx = \int \sec x \, \sec^2 x \, dx$$

Let $U = \sec x, \; dV = \sec^2 x \, dx$.
$\Rightarrow dU = \sec x \tan x \, dx, \; V = \tan x$.

Integration by Parts

Example. Evaluate the integral $I = \int \sec^3 x \, dx$.

Solution. Start using integration by parts:

$$I = \int \sec^3 x \, dx = \int \sec x \, \sec^2 x \, dx$$

Let $U = \sec x, \; dV = \sec^2 x \, dx$.
$\Rightarrow dU = \sec x \tan x \, dx, \; V = \tan x$.

$$= \sec x \tan x - \int \sec x \underbrace{\tan^2 x} \, dx$$

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Let $U = \sec x$, $dV = \sec^2 x \, dx$.
$\Rightarrow dU = \sec x \tan x \, dx$, $V = \tan x$.

$$= \sec x \tan x - \int \sec x \underbrace{\tan^2 x} \, dx$$

$\tan^2 x = \sec^2 x - 1$

Integration by Parts

Example. Evaluate the integral $I = \int \sec^3 x \, dx$.

Solution. Start using integration by parts:

$$I = \int \sec^3 x \, dx = \int \sec x \, \sec^2 x \, dx$$

Let $U = \sec x, \; dV = \sec^2 x \, dx$.
$\Rightarrow dU = \sec x \tan x \, dx, \; V = \tan x$.

$$= \sec x \tan x - \int \sec x \underbrace{\tan^2 x} \, dx$$

$\tan^2 x = \sec^2 x - 1$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

Integration by Parts

Example. Evaluate the integral $I = \displaystyle\int \sec^3 x \, dx$.

Solution. Start using integration by parts:

$$I = \int \sec^3 x \, dx = \int \sec x \, \sec^2 x \, dx$$

Let $U = \sec x,\ dV = \sec^2 x \, dx.$
$\Rightarrow dU = \sec x \tan x \, dx,\ V = \tan x.$

$$= \sec x \tan x - \int \sec x \underbrace{\tan^2 x} \, dx$$

$\tan^2 x = \sec^2 x - 1$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$= \sec x \tan x - I + \ln|\sec x + \tan x| + C.$$

Integration by Parts

Example. Evaluate the integral $I = \int \sec^3 x \, dx$.

Solution. Start using integration by parts:

$$I = \int \sec^3 x \, dx = \int \sec x \, \sec^2 x \, dx$$

Let $U = \sec x$, $dV = \sec^2 x \, dx$.
$\Rightarrow dU = \sec x \tan x \, dx$, $V = \tan x$.

$$= \sec x \tan x - \int \sec x \underbrace{\tan^2 x} \, dx$$

$\tan^2 x = \sec^2 x - 1$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$= \sec x \tan x - I + \ln|\sec x + \tan x| + C.$$

Therefore, we have $2I = \sec x \tan x + \ln|\sec x + \tan x| + C$. So

$$I = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C.$$

Integration by Parts

Example. Evaluate the integral $I = \int e^{ax} \cos bx \, dx$.

Solution. If either $a = 0$ or $b = 0$, the integral is easy to calculate, so let us assume that $a, b \neq 0$. We have

$$I = \int e^{ax} \cos bx \, dx$$

$U = e^{ax}$, $dV = \cos bx \, dx$.
$\Rightarrow dU = ae^{ax} \, dx$, $V = \frac{1}{b} \sin bx$.

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx$$

$U = e^{ax}$, $dV = \sin bx \, dx$.
$\Rightarrow dU = ae^{ax} \, dx$, $V = -\frac{1}{b} \cos bx$.

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \left(-\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx \right)$$

$$= \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} I \quad \Rightarrow \quad I = \frac{be^{ax} \sin bx + ae^{ax} \cos bx}{b^2 + a^2} + C$$

Example (A definite integral).

$$\int_1^e x^3(\ln x)^2\,dx$$

Let $U = (\ln x)^2,\ dV = x^3\,dx.$
$\Rightarrow dU = 2(\ln x)/x\,dx,\ V = x^4/4.$

$$= \frac{x^4}{4}(\ln x)^2\Big|_1^e - \frac{1}{2}\int_1^e x^3 \ln x\,dx$$

Let $U = \ln x,\ dV = x^3\,dx.$
$\Rightarrow dU = dx/x,\ V = x^4/4.$

$$= \frac{e^4}{4}(1^2) - 0 - \frac{1}{2}\left(\frac{x^4}{4}\ln x\Big|_1^e - \frac{1}{4}\int_1^e x^3 dx\right)$$

$$= \frac{e^4}{4} - \frac{e^4}{8} + \frac{1}{8}\frac{x^4}{4}\Big|_1^e$$

$$= \frac{e^4}{8} + \frac{1}{32}(e^4 - 1)$$

Integration by Parts

Example. Obtain and use a reduction formula to evaluate

$$I_n = \int_0^{\pi/2} \cos^n x \, dx \qquad (n = 0, 1, 2, 3, \ldots).$$

Solution. For $n = 0$, we have $I_0 = \int_0^{\pi/2} 1 \, dx = \pi/2$. For $n = 1$, we have $I_1 = \int_0^{\pi/2} \cos x \, dx = 1$.

Example. Obtain and use a reduction formula to evaluate

$$I_n = \int_0^{\pi/2} \cos^n x \, dx \qquad (n = 0, 1, 2, 3, \ldots).$$

Solution. For $n = 0$, we have $I_0 = \int_0^{\pi/2} 1 \, dx = \pi/2$. For $n = 1$, we have $I_1 = \int_0^{\pi/2} \cos x \, dx = 1$. For $n \geq 2$, using integration by parts, we get

$$I_n = \int_0^{\pi/2} \cos^{n-1} x \, \cos x \, dx$$

Integration by Parts

Example. Obtain and use a reduction formula to evaluate

$$I_n = \int_0^{\pi/2} \cos^n x \, dx \qquad (n = 0, 1, 2, 3, \ldots).$$

Solution. For $n = 0$, we have $I_0 = \int_0^{\pi/2} 1 \, dx = \pi/2$. For $n = 1$, we have $I_1 = \int_0^{\pi/2} \cos x \, dx = 1$. For $n \geq 2$, using integration by parts, we get

$$\begin{aligned} I_n &= \int_0^{\pi/2} \cos^{n-1} x \, \cos x \, dx \\ &= \sin x \cos^{n-1} x \Big|_0^{\pi/2} + (n-1) \int_0^{\pi/2} \cos^{n-2} x \sin^2 x \, dx \end{aligned}$$

Example. Obtain and use a reduction formula to evaluate

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Solution. For $n = 0$, we have $I_0 = \int_0^{\pi/2} 1 \, dx = \pi/2$. For $n = 1$, we have $I_1 = \int_0^{\pi/2} \cos x \, dx = 1$. For $n \geq 2$, using integration by parts, we get

$$\begin{aligned} I_n &= \int_0^{\pi/2} \cos^{n-1} x \, \cos x \, dx \\ &= \sin x \cos^{n-1} x \Big|_0^{\pi/2} + (n-1) \int_0^{\pi/2} \cos^{n-2} x \sin^2 x \, dx \\ &= 0 + (n-1) \int_0^{\pi/2} \cos^{n-2} x (1 - \cos^2 x) \, dx \end{aligned}$$

Integration by Parts

Example. Obtain and use a reduction formula to evaluate

$$I_n = \int_0^{\pi/2} \cos^n x \, dx \qquad (n = 0, 1, 2, 3, \ldots).$$

Solution. For $n = 0$, we have $I_0 = \int_0^{\pi/2} 1 \, dx = \pi/2$. For $n = 1$, we have $I_1 = \int_0^{\pi/2} \cos x \, dx = 1$. For $n \geq 2$, using integration by parts, we get

$$\begin{aligned} I_n &= \int_0^{\pi/2} \cos^{n-1} x \, \cos x \, dx \\ &= \sin x \cos^{n-1} x \Big|_0^{\pi/2} + (n-1) \int_0^{\pi/2} \cos^{n-2} x \sin^2 x \, dx \\ &= 0 + (n-1) \int_0^{\pi/2} \cos^{n-2} x (1 - \cos^2 x) \, dx \\ &= (n-1)\left(I_{n-2} - I_n\right) \end{aligned}$$

Integration by Parts

Example. Obtain and use a reduction formula to evaluate

$$I_n = \int_0^{\pi/2} \cos^n x \, dx \qquad (n = 0, 1, 2, 3, \ldots).$$

Solution. For $n = 0$, we have $I_0 = \int_0^{\pi/2} 1 \, dx = \pi/2$. For $n = 1$, we have $I_1 = \int_0^{\pi/2} \cos x \, dx = 1$. For $n \geq 2$, using integration by parts, we get

$$\begin{aligned}
I_n &= \int_0^{\pi/2} \cos^{n-1} x \, \cos x \, dx \\
&= \sin x \cos^{n-1} x \Big|_0^{\pi/2} + (n-1) \int_0^{\pi/2} \cos^{n-2} x \sin^2 x \, dx \\
&= 0 + (n-1) \int_0^{\pi/2} \cos^{n-2} x (1 - \cos^2 x) \, dx \\
&= (n-1) \left(I_{n-2} - I_n \right) \\
&\Rightarrow I_n = \frac{n-1}{n} I_{n-2}.
\end{aligned}$$

Integrals of Rational Functions

Integrals of Rational Functions

In this section, we are concerned with integrals of the form

$$I = \int \frac{P(x)}{Q(x)}\, dx$$

where $P(x)$ and $Q(x)$ are polynomials.

Integrals of Rational Functions

Example. Evaluate the integral $\int \frac{x^3+3x^2}{x^2+1}dx$.

Solution. The numerator has degree 3 and the denominator has degree 2, so we need to divide. We use long division:

$$\begin{array}{rrr|l} x^3 & +3x^2 & & x^2+1 \\ \hline -x^3 & & -x & x+3 \\ \hline & 3x^2 & -x & \\ & -3x^2 & -3 & \\ \hline & & -x-3 & \end{array} \qquad \Rightarrow \qquad \frac{x^3+3x^2}{x^2+1} = x+3-\frac{x+3}{x^2+1}.$$

Integrals of Rational Functions

Example. Evaluate the integral $\int \frac{x^3+3x^2}{x^2+1}dx$.

Solution. The numerator has degree 3 and the denominator has degree 2, so we need to divide. We use long division:

$$\begin{array}{rl} x^3 + 3x^2 \qquad\quad & \big| \; x^2+1 \\ -x^3 \qquad\quad -x & \big| \; x+3 \\ \hline 3x^2 - x & \\ -3x^2 \qquad -3 & \\ \hline -x-3 & \end{array} \qquad \Rightarrow \qquad \frac{x^3+3x^2}{x^2+1} = x+3-\frac{x+3}{x^2+1}.$$

Hence,

$$\begin{aligned} \int \frac{x^3+3x^2}{x^2+1}dx &= \int (x+3)dx - \int \frac{x}{x^2+1}dx - \int \frac{3}{x^2+1}dx \\ &= \frac{x^2}{2} + 3x - \frac{1}{2}\ln(x^2+1) - 3\tan^{-1}x + C. \end{aligned}$$

Example. Evaluate the integral $\int \frac{x}{2x-1}\,dx$.

Solution. The numerator and denominator have the same degree, 1, so division is again required. In this case the division can be carried out by manipulation of the integrand:

$$\frac{x}{2x-1} = \frac{1}{2}\frac{2x}{2x-1} = \frac{1}{2}\frac{2x-1+1}{2x-1} = \frac{1}{2}\left(1+\frac{1}{2x-1}\right),$$

a process that we call *short division*.

Example. Evaluate the integral $\int \frac{x}{2x-1}\,dx$.

Solution. The numerator and denominator have the same degree, 1, so division is again required. In this case the division can be carried out by manipulation of the integrand:

$$\frac{x}{2x-1} = \frac{1}{2}\frac{2x}{2x-1} = \frac{1}{2}\frac{2x-1+1}{2x-1} = \frac{1}{2}\left(1 + \frac{1}{2x-1}\right),$$

a process that we call *short division*. Thus,

$$\int \frac{x}{2x-1}\,dx = \frac{1}{2}\int \left(1 + \frac{1}{2x-1}\right) dx = \frac{x}{2} + \frac{1}{4}\ln|2x-1| + C.$$

Integrals of Rational Functions

The basic problem:

Evaluate $\int \frac{P(x)}{Q(x)}\, dx$, where the degree of $P(x)$ is less than the degree of $Q(x)$.

Integrals of Rational Functions

Linear and Quadratic Denominators

- Suppose that $Q(x)$ has degree 1. Thus, $Q(x) = ax + b$, where $a \neq 0$.

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The substitution $u = ax + b$ leads to

$$\int \frac{c}{ax+b}\,dx = \frac{c}{a}\int \frac{du}{u} = \frac{c}{a}\ln|u| + C = \frac{c}{a}\ln|ax+b| + C.$$

Let us now consider the case where $Q(x)$ is a quadratic polynomial of the form:

$$\int \frac{x}{x^2+a^2}\,dx, \quad \int \frac{x}{x^2-a^2}\,dx, \quad \int \frac{dx}{x^2+a^2}, \quad \text{and} \quad \int \frac{dx}{x^2-a^2}.$$

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The first three integrals can be readily evaluated using substitution:

- $\displaystyle\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln(x^2+a^2) + C,$

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- $\displaystyle\int \frac{dx}{x^2+a^2} = \int \frac{(1/a^2)dx}{1+(x/a)^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C.$

Integrals of Rational Functions

Linear and Quadratic Denominators

- $\dfrac{1}{x^2 - a^2} = \dfrac{1}{(x-a)(x+a)} = \dfrac{A}{x-a} + \dfrac{B}{x+a} = \dfrac{Ax + Aa + Bx - Ba}{x^2 - a^2},$

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$$\begin{aligned} A + B &= 0 \quad \text{the coefficient of } x \\ Aa - Ba &= 1 \quad \text{the constant term.} \end{aligned} \longrightarrow \boxed{\begin{aligned} A &= 1/(2a) \\ B &= -1/(2a). \end{aligned}}$$

Integrals of Rational Functions

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$$\begin{aligned} \int \frac{dx}{x^2-a^2} &= \frac{1}{2a}\int \frac{dx}{x-a} - \frac{1}{2a}\int \frac{dx}{x+a} \\ &= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C \\ &= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C. \end{aligned}$$

Suppose

$$Q(x) = (x - a_1)(x - a_2) \cdots (x - a_n),$$

where $a_i \neq a_j$ if $i \neq j$, $1 \leq i, j \leq n$. If $P(x)$ is a polynomial of degree smaller than n, then $P(x)/Q(x)$ has a **partial fraction decomposition** of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_n}{x - a_n},$$

for certain values of the constants $A_1, A_2, \ldots, A_n$.

Integrals of Rational Functions

Example. Evaluate the integral $\int \frac{x+4}{x^2-5x+6}dx$.

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$$\frac{x+4}{x^2-5x+6} = \frac{x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

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METHOD I.

$$\frac{x+4}{(x-2)(x-3)} = \frac{Ax-3A+Bx-2B}{(x-2)(x-3)} \Rightarrow \left.\begin{aligned} A+B &= 1 \\ -3A-2B &= 4 \end{aligned}\right\} \Rightarrow A=-6,\ B=7.$$

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METHOD II.

$$A = \left.\frac{x+4}{x-3}\right|_{x=2} = -6 \quad \text{and} \quad B = \left.\frac{x+4}{x-2}\right|_{x=3} = 7.$$

Integrals of Rational Functions

Example. Evaluate the integral $\displaystyle\int \frac{x+4}{x^2-5x+6}\,dx$.

Solution.

$$\begin{aligned}\int \frac{x+4}{x^2-5x+6}\,dx &= -6\int \frac{1}{x-2}\,dx + 7\int \frac{1}{x-3}\,dx \\ &= -6\ln|x-2| + 7\ln|x-3| + C.\end{aligned}$$

Integrals of Rational Functions

Example. Evaluate $\int \frac{x^3+2}{x^3-x}dx$.

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Solution. $\deg(P(x)) = \deg(Q(x)) = 3$, so we first use long division or short division:

$$I = \int \frac{x^3 - x + x + 2}{x^3 - x}dx = \int \left(1 + \frac{x+2}{x^3-x}\right)dx = x + \int \frac{x+2}{x^3-x}dx.$$

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Then we find that

$$A = \left.\frac{x+2}{(x-1)(x+1)}\right|_{x=0} = -2, \quad B = \left.\frac{x+2}{x(x+1)}\right|_{x=1} = \frac{3}{2},$$
$$C = \left.\frac{x+2}{x(x-1)}\right|_{x=-1} = \frac{1}{2}.$$

Example. Evaluate $\int \frac{x^3+2}{x^3-x}dx$.

Solution.

$$I = x - 2\int \frac{1}{x}dx + \frac{3}{2}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{x+1}dx$$
$$= x - 2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C.$$

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Integrals of Rational Functions

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Integrals of Rational Functions

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$$\begin{aligned} \int \frac{2+3x+x^2}{x(x^2+1)}dx &= 2\int \frac{1}{x}dx - \int \frac{x}{x^2+1}dx + 3\int \frac{1}{x^2+1}dx \\ &= 2\ln|x| - \frac{1}{2}\ln(x^2+1) + 3\tan^{-1}x + C. \end{aligned}$$

Example. Evaluate $\displaystyle\int \frac{1}{x^3+1}\,dx$.

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Here $Q(x) = x^3 + 1 = (x+1)(x^2 - x + 1)$. The latter factor has no real roots, so it has no real linear subfactors. We have

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$
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Example. Evaluate $\int \frac{1}{x^3+1}\,dx$.

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$$I = \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx$$

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$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{u}{u^2 + \frac{3}{4}}\,du + \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\,du$$

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$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(u^2+\frac{3}{4}\right) + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) + C$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(x^2-x+1\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C.$$

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Solution.

The appropriate partial fraction decomposition here is

$$\begin{aligned} \frac{1}{x(x-1)^2} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\ &= \frac{A(x^2-2x+1) + B(x^2-x) + Cx}{x(x-1)^2}. \end{aligned}$$

Equating coefficients of x^2, x, and 1 in the numerators of both sides, we get

$$\left.\begin{array}{rcll} A+B &=& 0 & \text{(coefficient of } x^2) \\ -2A-B+C &=& 0 & \text{(coefficient of } x) \\ A &=& 1 & \text{(constant term)} \end{array}\right\}$$

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Solution.

Using $A = 1,\ B = -1,\ C = 1$, we have

$$\begin{aligned}
\int \frac{1}{x(x-1)^2}\,dx &= \int \left(\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\right) dx \\
&= \int \frac{1}{x}\,dx - \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx \\
&= \ln|x| - \ln|x-1| - \frac{1}{x-1} + C \\
&= \ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C.
\end{aligned}$$

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$$\left.\begin{array}{rcl} 4A+2B &=& 0 \\ 2C &=& 0 \\ 4A+B+D &=& 1 \\ C+E &=& 0 \\ A &=& 2 \end{array}\right\}$$

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$$\left.\begin{array}{rcl} 4A+2B &=& 0 \\ 2C &=& 0 \\ 4A+B+D &=& 1 \\ C+E &=& 0 \\ A &=& 2 \end{array}\right\} \quad \Rightarrow \quad \begin{array}{c} A=2, \\ B=-4, \\ C=0, \\ D=-3, \\ E=0. \end{array}$$

Example. Evaluate $\displaystyle\int \frac{x^2+2}{4x^5+4x^3+x}\,dx$.

Solution.

$$I = 2\int \frac{dx}{x} - 4\int \frac{x\,dx}{2x^2+1} - 3\int \frac{x\,dx}{(2x^2+1)^2}$$

Example. Evaluate $\displaystyle\int \frac{x^2+2}{4x^5+4x^3+x}\,dx.$

Solution.

$$\begin{aligned} I &= 2\int \frac{dx}{x} - 4\int \frac{x\,dx}{2x^2+1} - 3\int \frac{x\,dx}{(2x^2+1)^2} \\ &= 2\ln|x| - \int \frac{du}{u} - \frac{3}{4}\int \frac{du}{u^2} \end{aligned}$$

Let $u = 2x^2+1$.
Then $du = 4x\,dx$.

Integrals of Rational Functions

Denominators with Repeated Factors

Example. Evaluate $\displaystyle\int \frac{x^2+2}{4x^5+4x^3+x}\,dx.$

Solution.

$$\begin{aligned} I &= 2\int \frac{dx}{x} - 4\int \frac{x\,dx}{2x^2+1} - 3\int \frac{x\,dx}{(2x^2+1)^2} \\ &= 2\ln|x| - \int \frac{du}{u} - \frac{3}{4}\int \frac{du}{u^2} \\ &= 2\ln|x| - \ln|u| + \frac{3}{4u} + C \end{aligned}$$

Let $u = 2x^2 + 1$.
Then $du = 4x\,dx$.

Example. Evaluate $\displaystyle\int \frac{x^2+2}{4x^5+4x^3+x}\,dx$.

Solution.

$$\begin{aligned} I &= 2\int \frac{dx}{x} - 4\int \frac{x\,dx}{2x^2+1} - 3\int \frac{x\,dx}{(2x^2+1)^2} \\ &= 2\ln|x| - \int \frac{du}{u} - \frac{3}{4}\int \frac{du}{u^2} \\ &= 2\ln|x| - \ln|u| + \frac{3}{4u} + C \\ &= \ln\left(\frac{x^2}{2x^2+1}\right) + \frac{3}{4(2x^2+1)} + C. \end{aligned}$$

Let $u = 2x^2+1$.
Then $du = 4x\,dx$.

Example. Evaluate $\displaystyle\int \frac{x^2+2}{4x^5+4x^3+x}\,dx$.

Solution.

$$\begin{aligned}
I &= 2\int \frac{dx}{x} - 4\int \frac{x\,dx}{2x^2+1} - 3\int \frac{x\,dx}{(2x^2+1)^2} \\
&= 2\ln|x| - \int \frac{du}{u} - \frac{3}{4}\int \frac{du}{u^2} \\
&= 2\ln|x| - \ln|u| + \frac{3}{4u} + C \\
&= \ln\left(\frac{x^2}{2x^2+1}\right) + \frac{3}{4(2x^2+1)} + C.
\end{aligned}$$

Let $u = 2x^2+1$.
Then $du = 4x\,dx$.