MAT123 MATHEMATICS I

Lecture 20: Techniques of Integration (Inverse Substitutions), Improper Integrals

Outline

Inverse Substitutions

Inverse Substitutions

$$\int_a^b f(x)dx \xrightarrow{x=g(u)} \int_{x=a}^{x=b} f(g(u))g'(u)du.$$

Inverse Substitutions

The Inverse Trigonometric Substitutions

Three very useful inverse trigonometric substitutions are

$$x = a\sin\theta, \quad x = a\tan\theta, \quad x = a\sec\theta.$$

The inverse sine substitution: Integrals involving $\sqrt{a^2 - x^2}$ (where $a > 0$) can frequently be reduced to a simpler form by the substitution

$$x = a\sin\theta \qquad \Rightarrow \qquad dx = a\cos\theta\, d\theta.$$

Observe that $\sqrt{a^2 - x^2}$ makes sense only if $-a \le x \le a$, so the range of θ is $-\pi/2 \le \theta \le \pi/2$. Since $\cos\theta \ge 0$ in this range, we have

$$\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta.$$

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{(5-x^2)^{3/2}}\,dx$.

Solution.

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{(5-x^2)^{3/2}}\,dx$.

Solution.

$$\int \frac{1}{(5-x^2)^{3/2}}\,dx$$

Let $x = \sqrt{5}\,\sin\theta$.
Then $dx = \sqrt{5}\cos\theta\,d\theta$.

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\int \frac{1}{(5-x^2)^{3/2}}\,dx$.

Solution.

$$\int \frac{1}{(5-x^2)^{3/2}}\,dx$$

$$= \int \frac{\sqrt{5}\cos\theta\,d\theta}{5^{3/2}\cos^3\theta}$$

Let $x = \sqrt{5}\,\sin\theta$.
Then $dx = \sqrt{5}\cos\theta\,d\theta$.

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{(5-x^2)^{3/2}}\,dx$.

Solution.

Let $x = \sqrt{5}\,\sin\theta$.
Then $dx = \sqrt{5}\cos\theta\,d\theta$.

$$\begin{aligned}
&\int \frac{1}{(5-x^2)^{3/2}}\,dx \\
&\quad= \int \frac{\sqrt{5}\cos\theta\,d\theta}{5^{3/2}\cos^3\theta} \\
&\quad= \frac{1}{5}\int \sec^2\theta\,d\theta \\
&\quad= \frac{1}{5}\tan\theta + C
\end{aligned}$$

Inverse Substitutions

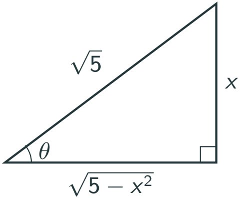
The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{(5-x^2)^{3/2}}\,dx$.

Solution.

$$\begin{aligned}\int \frac{1}{(5-x^2)^{3/2}}\,dx &= \int \frac{\sqrt{5}\cos\theta\,d\theta}{5^{3/2}\cos^3\theta} \\ &= \frac{1}{5}\int \sec^2\theta\,d\theta \\ &= \frac{1}{5}\tan\theta + C\end{aligned}$$

Let $x = \sqrt{5}\sin\theta$.
Then $dx = \sqrt{5}\cos\theta\,d\theta$.



Inverse Substitutions

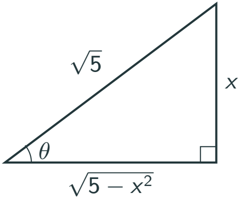
The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{(5-x^2)^{3/2}}\,dx$.

Solution.

$$\begin{aligned}
\int \frac{1}{(5-x^2)^{3/2}}\,dx \\
&= \int \frac{\sqrt{5}\cos\theta\,d\theta}{5^{3/2}\cos^3\theta} \\
&= \frac{1}{5}\int \sec^2\theta\,d\theta \\
&= \frac{1}{5}\tan\theta + C \\
&= \frac{1}{5}\,\frac{x}{\sqrt{5-x^2}} + C.
\end{aligned}$$

Let $x = \sqrt{5}\,\sin\theta$.
Then $dx = \sqrt{5}\cos\theta\,d\theta$.



Inverse Substitutions

The Inverse Trigonometric Substitutions

The inverse tangent substitution: Integrals involving $\sqrt{x^2 + a^2}$ or $\dfrac{1}{\sqrt{x^2 + a^2}}$ (where $a > 0$) are often simplified by the substitution

$$x = a \tan\theta \qquad \Rightarrow \qquad dx = a \sec^2\theta \, d\theta.$$

Since x can take any real value, we have $-\pi/2 < \theta < \pi/2$, so $\sec\theta > 0$ and

$$\sqrt{x^2 + a^2} = \sqrt{a^2 \tan^2\theta + a^2} = \sqrt{a^2(\tan^2\theta + 1)} = a \sec\theta.$$

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\int \frac{1}{(1+9x^2)^2}\,dx$.

Solution.

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\int \frac{1}{(1+9x^2)^2}\,dx$.

Solution.

$$\int \frac{1}{(1+9x^2)^2}\,dx$$

Let $3x = \tan\theta$.
Then $3\,dx = \sec^2\theta\,d\theta$.

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{(1+9x^2)^2}\,dx$.

Solution.

Let $3x = \tan\theta$.
Then $3\,dx = \sec^2\theta\,d\theta$.

$$\int \frac{1}{(1+9x^2)^2}\,dx$$

$$= \frac{1}{3}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \frac{1}{3}\int \cos^2\theta\,d\theta$$

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{(1+9x^2)^2}\,dx$.

Solution.

Let $3x = \tan\theta$.
Then $3\,dx = \sec^2\theta\,d\theta$.

$$\begin{aligned}
&\int \frac{1}{(1+9x^2)^2}\,dx \\
&\quad = \frac{1}{3}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta \\
&\quad = \frac{1}{3}\int \cos^2\theta\,d\theta \\
&\quad = \frac{1}{6}\big(\theta + \sin\theta\cos\theta\big) + C
\end{aligned}$$

Inverse Substitutions

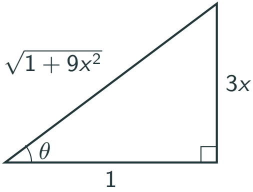
The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{(1+9x^2)^2}\,dx$.

Solution.

$$\begin{aligned}
&\int \frac{1}{(1+9x^2)^2}\,dx \\
&\quad = \frac{1}{3}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta \\
&\quad = \frac{1}{3}\int \cos^2\theta\,d\theta \\
&\quad = \frac{1}{6}\big(\theta + \sin\theta\cos\theta\big) + C
\end{aligned}$$

Let $3x = \tan\theta$.
Then $3\,dx = \sec^2\theta\,d\theta$.



Inverse Substitutions

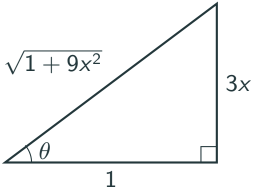
The Inverse Trigonometric Substitutions

Example. Evaluate $\int \frac{1}{(1+9x^2)^2}\,dx$.

Solution.

Let $3x = \tan\theta$.
Then $3\,dx = \sec^2\theta\,d\theta$.

$$\begin{aligned}
&\int \frac{1}{(1+9x^2)^2}\,dx \\
&\quad = \frac{1}{3}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta \\
&\quad = \frac{1}{3}\int \cos^2\theta\,d\theta \\
&\quad = \frac{1}{6}\big(\theta + \sin\theta\cos\theta\big) + C \\
&\quad = \frac{1}{6}\tan^{-1}(3x) + \frac{1}{6}\,\frac{3x}{\sqrt{1+9x^2}}\,\frac{1}{\sqrt{1+9x^2}} + C \\
&\quad = \frac{1}{6}\tan^{-1}(3x) + \frac{1}{2}\,\frac{x}{1+9x^2} + C.
\end{aligned}$$



Inverse Substitutions

The Inverse Trigonometric Substitutions

The inverse secant substitution: Integrals involving $\sqrt{x^2 - a^2}$ (where $a > 0$) can frequently be simplified by the substitution

$$x = a \sec\theta \qquad \Rightarrow \qquad dx = a \sec\theta \tan\theta \, d\theta.$$

$$\sqrt{x^2 - a^2} = a\sqrt{\sec^2\theta - 1} = a\sqrt{\tan^2\theta} = a|\tan\theta|.$$

Observe that $\sqrt{x^2 - a^2}$ makes sense only if $x \geq a$ or $x \leq -a$.

- If $x \geq a$, then $0 \leq \theta = \sec^{-1}\frac{x}{a} = \arccos\frac{a}{x} < \frac{\pi}{2}$, and $\tan\theta \geq 0$.
- If $x \leq -a$, then $\frac{\pi}{2} < \theta = \sec^{-1}\frac{x}{a} = \arccos\frac{a}{x} \leq \pi$, and $\tan\theta \leq 0$.

In the first case $\sqrt{x^2 - a^2} = a\tan\theta$; in the second case $\sqrt{x^2 - a^2} = -a\tan\theta$.

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$, where $a > 0$.

Solution.

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx$, where $a>0$.

Solution.

Case 1: Assume $x \geq a$. Set $x = a\sec\theta$, so
$dx = a\sec\theta\tan\theta\,d\theta$ and

$$\sqrt{x^2-a^2} = a\tan\theta.$$

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$, where $a > 0$.

Solution.

Case 1: Assume $x \geq a$. Set $x = a\sec\theta$, so $dx = a\sec\theta\tan\theta\, d\theta$ and

$$\sqrt{x^2 - a^2} = a\tan\theta.$$

$$\begin{aligned}\implies I &= \int \sec\theta\, d\theta \\ &= \ln|\sec\theta + \tan\theta| + C\end{aligned}$$

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx$, where $a > 0$.

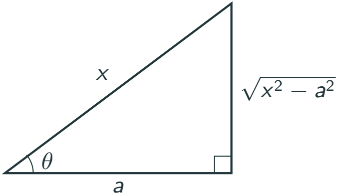
Solution.

Case 1: Assume $x \geq a$. Set $x = a\sec\theta$, so $dx = a\sec\theta\tan\theta\,d\theta$ and

$$\sqrt{x^2 - a^2} = a\tan\theta.$$

$$\implies I = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$



Inverse Substitutions

The Inverse Trigonometric Substitutions

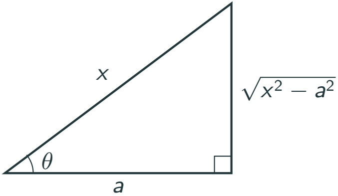
Example. Evaluate $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$, where $a > 0$.

Solution.

Case 1: Assume $x \geq a$. Set $x = a\sec\theta$, so $dx = a\sec\theta\tan\theta\, d\theta$ and

$$\sqrt{x^2 - a^2} = a\tan\theta.$$

$$\begin{aligned}\implies I &= \int \sec\theta\, d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C \\ &= \ln|x + \sqrt{x^2 - a^2}| + C_1,\ \ C_1 = C - \ln a.\end{aligned}$$



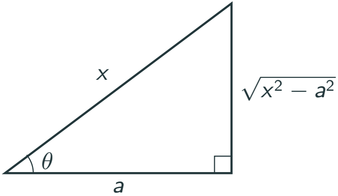
Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\,dx$, where $a > 0$.

Solution.

Case 2: Now assume $x \leq -a$. Let $u = -x$, so $u \geq a$ and $du = -dx$.



Inverse Substitutions

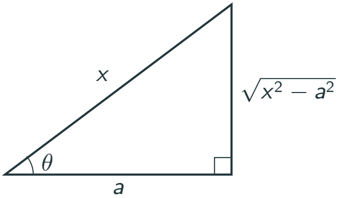
The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\,dx$, where $a > 0$.

Solution.

Case 2: Now assume $x \le -a$. Let $u = -x$, so $u \ge a$ and $du = -dx$.

$$I = -\int \frac{du}{\sqrt{u^2 - a^2}} = -\ln\left|u + \sqrt{u^2 - a^2}\right| + C_1$$



Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$, where $a > 0$.

Solution.

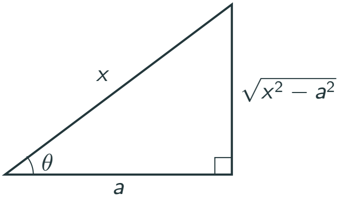
Case 2: Now assume $x \le -a$. Let $u = -x$, so $u \ge a$ and $du = -dx$.

$$I = -\int \frac{du}{\sqrt{u^2 - a^2}} = -\ln\left|u + \sqrt{u^2 - a^2}\right| + C_1$$

$$= \ln\left|\frac{1}{-x + \sqrt{x^2 - a^2}} \cdot \frac{x + \sqrt{x^2 - a^2}}{x + \sqrt{x^2 - a^2}}\right| + C_1$$

$$= \ln\left|\frac{x + \sqrt{x^2 - a^2}}{-a^2}\right| + C_1$$

$$= \ln\left|x + \sqrt{x^2 - a^2}\right| + C_2,\ \ C_2 = C_1 - 2\ln a.$$



Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\,dx$, where $a > 0$.

Solution.

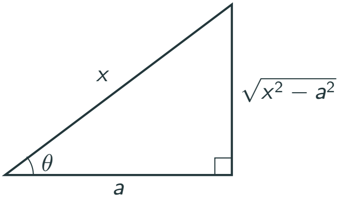
Case 2: Now assume $x \leq -a$. Let $u = -x$, so $u \geq a$ and $du = -dx$.

$$I = -\int \frac{du}{\sqrt{u^2 - a^2}} = -\ln\left|u + \sqrt{u^2 - a^2}\right| + C_1$$

$$= \ln\left|\frac{1}{-x + \sqrt{x^2 - a^2}} \cdot \frac{x + \sqrt{x^2 - a^2}}{x + \sqrt{x^2 - a^2}}\right| + C_1$$

$$= \ln\left|\frac{x + \sqrt{x^2 - a^2}}{-a^2}\right| + C_1$$

$$= \ln\left|x + \sqrt{x^2 - a^2}\right| + C_2,\ \ C_2 = C_1 - 2\ln a.$$



Therefore, in either case:

$$\boxed{I = \ln\left|x + \sqrt{x^2 - a^2}\right| + C}$$

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\int \frac{1}{\sqrt{2x - x^2}} dx$.

Solution.

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\int \frac{1}{\sqrt{2x - x^2}} dx$.

Solution.

$$\int \frac{1}{\sqrt{2x - x^2}} dx = \int \frac{dx}{\sqrt{1 - (1 - 2x + x^2)}}$$

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx$.

Solution.

$$\begin{aligned}\int \frac{1}{\sqrt{2x-x^2}}\,dx &= \int \frac{dx}{\sqrt{1-(1-2x+x^2)}}\\ &= \int \frac{dx}{\sqrt{1-(x-1)^2}}\end{aligned}$$

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\int \frac{1}{\sqrt{2x-x^2}}\,dx$.

Solution.

$$\int \frac{1}{\sqrt{2x-x^2}}\,dx = \int \frac{dx}{\sqrt{1-(1-2x+x^2)}}$$
$$= \int \frac{dx}{\sqrt{1-(x-1)^2}}$$

Let $u = x - 1$.
Then $du = dx$.

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\int \frac{1}{\sqrt{2x - x^2}}\,dx$.

Solution.

$$\begin{aligned}\int \frac{1}{\sqrt{2x - x^2}}\,dx &= \int \frac{dx}{\sqrt{1 - (1 - 2x + x^2)}} \\ &= \int \frac{dx}{\sqrt{1 - (x-1)^2}} \\ &= \int \frac{du}{\sqrt{1 - u^2}}\end{aligned}$$

Let $u = x - 1$.
Then $du = dx$.

Inverse Substitutions

The Inverse Trigonometric Substitutions

Example. Evaluate $\int \frac{1}{\sqrt{2x - x^2}} dx$.

Solution.

$$\begin{aligned}\int \frac{1}{\sqrt{2x - x^2}} dx &= \int \frac{dx}{\sqrt{1 - (1 - 2x + x^2)}} \\ &= \int \frac{dx}{\sqrt{1 - (x-1)^2}} \\ &= \int \frac{du}{\sqrt{1 - u^2}} \\ &= \sin^{-1} u + C = \sin^{-1}(x-1) + C.\end{aligned}$$

Let $u = x - 1$.
Then $du = dx$.

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $I = \int \frac{1}{1+\sqrt{2x}}\, dx$.

Solution.

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $I = \int \frac{1}{1+\sqrt{2x}}\,dx$.

Solution. Let $2x = u^2$, so that $2dx = 2u\,du$.

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $I = \displaystyle\int \frac{1}{1+\sqrt{2x}}\,dx$.

Solution. Let $2x = u^2$, so that $2dx = 2u\,du$. Then

$$I = \int \frac{u}{1+u}\,du$$

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $I = \displaystyle\int \frac{1}{1+\sqrt{2x}}\,dx$.

Solution. Let $2x = u^2$, so that $2dx = 2u\,du$. Then

$$\begin{aligned} I &= \int \frac{u}{1+u}\,du \\ &= \int \frac{1+u-1}{1+u}\,du \\ &= \int \left(1 - \frac{1}{1+u}\right) du \end{aligned}$$

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $I = \int \frac{1}{1+\sqrt{2x}}\,dx$.

Solution. Let $2x = u^2$, so that $2dx = 2u\,du$. Then

$$\begin{aligned} I &= \int \frac{u}{1+u}\,du \\ &= \int \frac{1+u-1}{1+u}\,du \\ &= \int \left(1 - \frac{1}{1+u}\right) du \end{aligned}$$

Let $v = 1+u$.
Then $dv = du$.

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $I = \int \frac{1}{1+\sqrt{2x}}\,dx$.

Solution. Let $2x = u^2$, so that $2dx = 2u\,du$. Then

$$\begin{aligned} I &= \int \frac{u}{1+u}\,du \\ &= \int \frac{1+u-1}{1+u}\,du \\ &= \int \left(1 - \frac{1}{1+u}\right) du \\ &= u - \int \frac{dv}{v} = u - \ln|v| + C \end{aligned}$$

Let $v = 1 + u$.
Then $dv = du$.

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $I = \int \frac{1}{1+\sqrt{2x}}\,dx$.

Solution. Let $2x = u^2$, so that $2dx = 2u\,du$. Then

$$\begin{aligned} I &= \int \frac{u}{1+u}\,du \\ &= \int \frac{1+u-1}{1+u}\,du \\ &= \int \left(1 - \frac{1}{1+u}\right) du \\ &= u - \int \frac{dv}{v} = u - \ln|v| + C \\ &= \sqrt{2x} - \ln(1+\sqrt{2x}) + C. \end{aligned}$$

Let $v = 1 + u$.
Then $dv = du$.

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $\int_{-1/3}^{2} \frac{x}{\sqrt[3]{3x+2}} dx$.

Solution.

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $\displaystyle\int_{-1/3}^{2} \frac{x}{\sqrt[3]{3x+2}}\,dx$.

Solution.

$$\int_{-1/3}^{2} \frac{x}{\sqrt[3]{3x+2}}\,dx$$

Let $3x+2=u^3$.
Then $3dx=3u^2\,du$.
$x=-1/3 \Rightarrow u=1$,
$x=2 \Rightarrow u=2$.

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $\displaystyle\int_{-1/3}^{2} \frac{x}{\sqrt[3]{3x+2}}\,dx$.

Solution.

$$\begin{aligned}
&\int_{-1/3}^{2} \frac{x}{\sqrt[3]{3x+2}}\,dx \\
&= \int_{1}^{2} \frac{u^3-2}{3u}\,u^2\,du \\
&= \frac{1}{3}\int_{1}^{2} \left(u^4-2u\right)du
\end{aligned}$$

Let $3x+2=u^3$.
Then $3dx = 3u^2\,du$.
$x=-1/3 \Rightarrow u=1$,
$x=2 \Rightarrow u=2$.

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $\displaystyle\int_{-1/3}^{2} \frac{x}{\sqrt[3]{3x+2}}\,dx.$

Solution.

Let $3x+2=u^3$.
Then $3dx = 3u^2\,du$.
$x=-1/3 \Rightarrow u=1$,
$x=2 \Rightarrow u=2$.

$$\begin{aligned}
&\int_{-1/3}^{2} \frac{x}{\sqrt[3]{3x+2}}\,dx \\
&= \int_{1}^{2} \frac{u^3-2}{3u}\,u^2\,du \\
&= \frac{1}{3}\int_{1}^{2} \left(u^4-2u\right) du \\
&= \frac{1}{3}\left(\frac{u^5}{5}-u^2\right)\Bigg|_{1}^{2} \\
&= \frac{16}{15}.
\end{aligned}$$

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx$.

Solution.

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx$.

Solution.

$$\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx$$

Let $x = u^6$.
Then $dx = 6u^5\,du$.

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $\displaystyle\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx$.

Solution.

$$\begin{aligned}&\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx\\ &\quad = 6\int \frac{u^5\,du}{u^3(1+u^2)} = 6\int \frac{u^2\,du}{1+u^2}\end{aligned}$$

Let $x = u^6$.
Then $dx = 6u^5\,du$.

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx$.

Solution.

$$\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx$$

$$= 6\int \frac{u^5\,du}{u^3(1+u^2)} = 6\int \frac{u^2\,du}{1+u^2}$$

$$= 6\int \left(1 - \frac{1}{1+u^2}\right)du$$

Let $x = u^6$.
Then $dx = 6u^5\,du$.

Inverse Substitutions

Other Inverse Substitutions

Example. Evaluate $\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx$.

Solution.

Let $x = u^6$.
Then $dx = 6u^5\,du$.

$$\begin{aligned}
&\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx \\
&\quad = 6\int \frac{u^5\,du}{u^3(1+u^2)} = 6\int \frac{u^2\,du}{1+u^2} \\
&\quad = 6\int \left(1 - \frac{1}{1+u^2}\right) du \\
&\quad = 6(u - \tan^{-1} u) + C = 6(x^{1/6} - \tan^{-1}(x^{1/6})) + C.
\end{aligned}$$

Inverse Substitutions

The $\tan(\theta/2)$ Substitution

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$$dx = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$$

$$\boxed{d\theta = 2\cos^2\frac{\theta}{2}\,dx = \frac{2\,dx}{1+x^2}}$$

Inverse Substitutions

The $\tan(\theta/2)$ Substitution

In summary:

If $x = \tan(\theta/2)$, then

$$\boxed{\cos\theta = \frac{1-x^2}{1+x^2}, \quad \sin\theta = \frac{2x}{1+x^2}, \quad d\theta = \frac{2\,dx}{1+x^2}.}$$

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Example.
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Inverse Substitutions

The $\tan(\theta/2)$ Substitution

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Improper Integrals

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$$I = \int_a^b f(x)dx.$$

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Integrals satisfying (i) are called **improper integrals of type I**,

Integrals satisfying (ii) are called **improper integrals of type II**.

Improper Integrals

Improper Integrals of Type I

Example. Find the area of the region A lying under the curve $y = 1/x^2$ and above the x-axis to the right of the line $x = 1$.

Solution.

Improper Integrals

Improper Integrals of Type I

Example. Find the area of the region A lying under the curve $y = 1/x^2$ and above the x-axis to the right of the line $x = 1$.

Solution. We would like to calculate the area with an integral

$$A = \int_1^\infty \frac{1}{x^2}\,dx,$$

which is improper of type I, since its interval of integration is infinite.

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Improper Integrals of Type I

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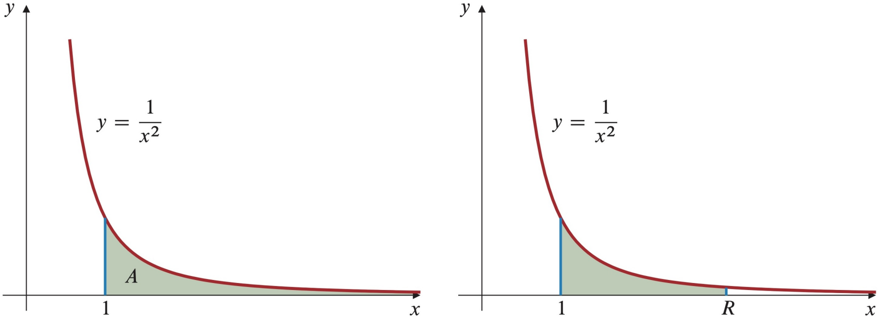

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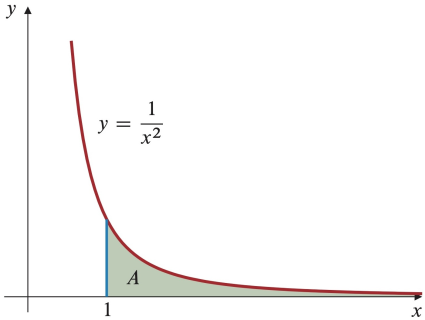

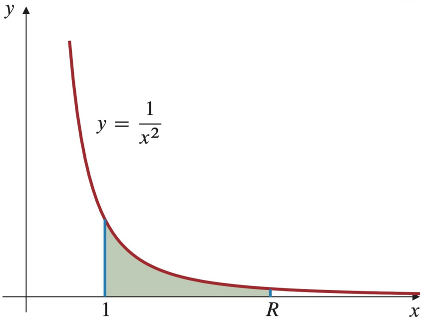

$$A = \int_1^\infty \frac{dx}{x^2} \quad = \lim_{R\to\infty} \int_1^R \frac{dx}{x^2}$$

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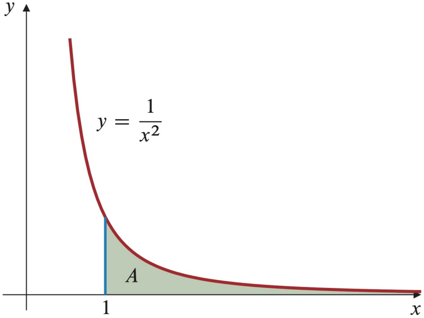

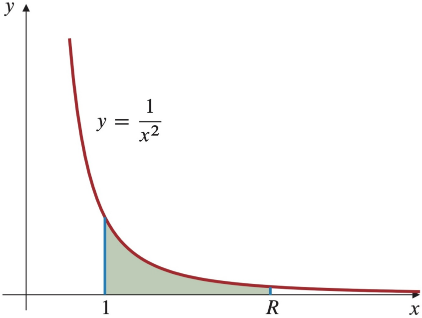

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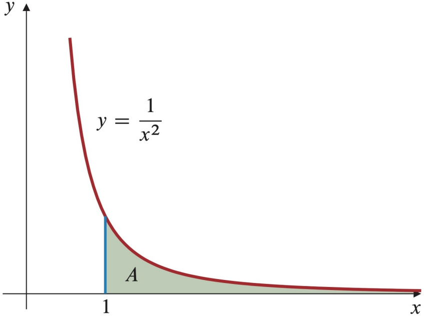

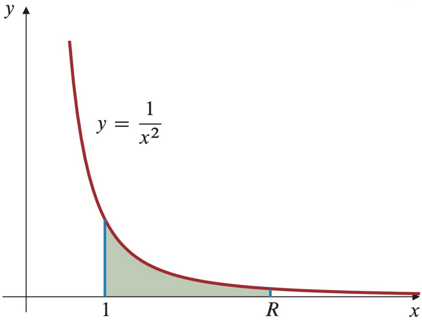

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Improper Integrals

Improper Integrals of Type I

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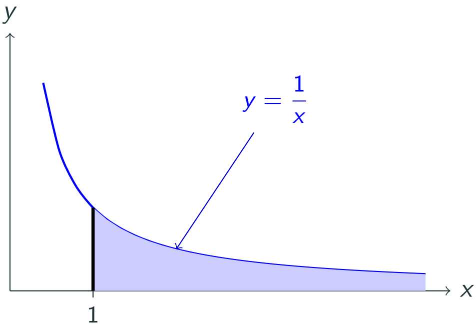
Solution.



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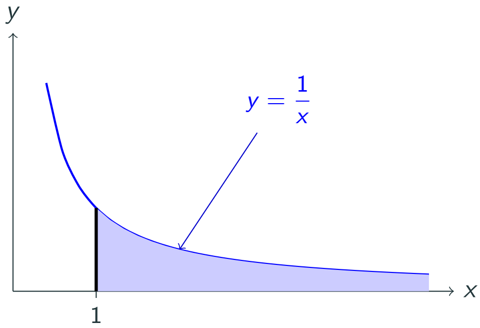

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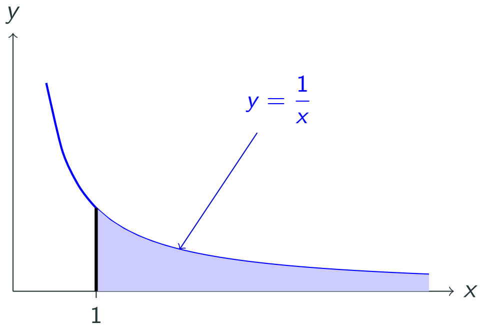

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$$\begin{aligned} A &= \int_1^\infty \frac{dx}{x} = \lim_{R\to\infty} \int_1^R \frac{dx}{x} \\ &= \lim_{R\to\infty} [\ln|x|]_1^R \end{aligned}$$

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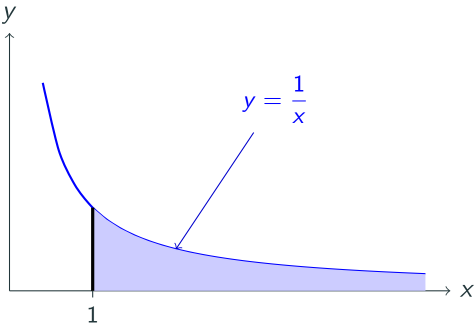

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&= \lim_{R\to\infty} (\ln R - \ln 1) = \infty
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$$

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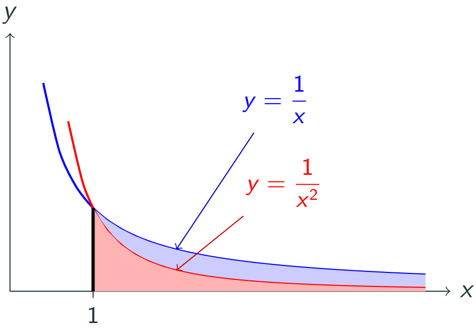

For comparison, consider the area under $y = 1/x^2$ on the same interval which is equal to 1. Thus $y = 1/x$ has an *infinite* area over $[1, \infty)$, while $y = 1/x^2$ has a *finite* area, even though both curves approach 0 as $x \to \infty$.

Improper Integrals

Improper Integrals of Type I

Definition. *Improper Integrals of Type I*

If f is continuous on $[a, \infty)$, we define the improper integral of f over $[a, \infty)$ as a limit of proper integrals:

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

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Similarly, if f is continuous on $(-\infty, b]$, we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{R \to -\infty} \int_R^b f(x)\,dx.$$

In either case, if the limit exists (is a finite number), we say that the improper integral **converges**; if the limit does not exist, we say that the improper integral **diverges**. If the limit is ∞ or $-\infty$, we say that the improper integral **diverges to infinity** or **diverges to negative infinity**, respectively.

Improper Integrals

Improper Integrals of Type I

The integral $\int_{-\infty}^{\infty} f(x)\,dx$ is, for f continuous on the real line, improper of type I at both endpoints. We break it into two separate integrals:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx.$$

The integral on the left converges if and only if both integrals on the right converge.

Improper Integrals

Improper Integrals of Type I

Example. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Solution.

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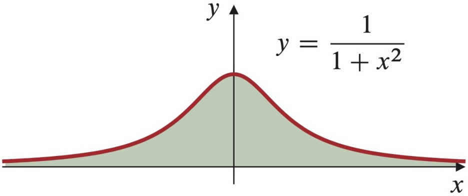


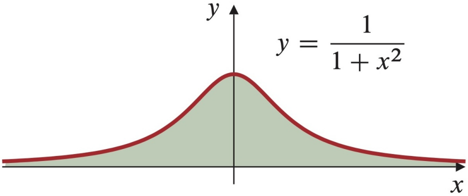
Improper Integrals

Improper Integrals of Type I

Example. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Solution. By the (even) symmetry of the integrand, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx \\ &= 2\int_{0}^{\infty} \frac{1}{1+x^2}\,dx\end{aligned}$$



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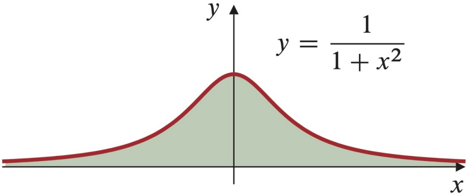

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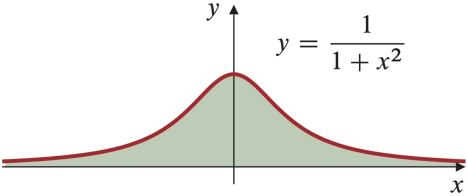

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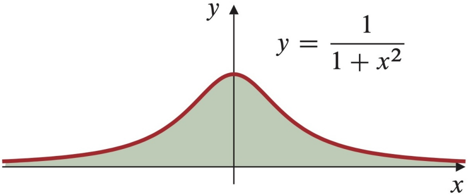

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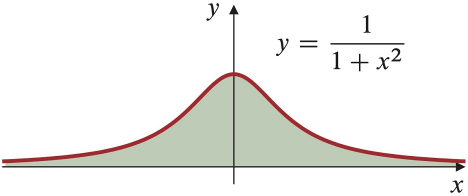
&= 2\int_{0}^{\infty} \frac{1}{1+x^2}\,dx \\
&= 2\lim_{R\to\infty} \int_{0}^{R} \frac{1}{1+x^2}\,dx \\
&= 2\lim_{R\to\infty} \left[\tan^{-1}(x)\right]_0^R \\
&= 2\lim_{R\to\infty} \left(\tan^{-1}(R) - \tan^{-1}(0)\right) \\
&= \pi
\end{aligned}$$

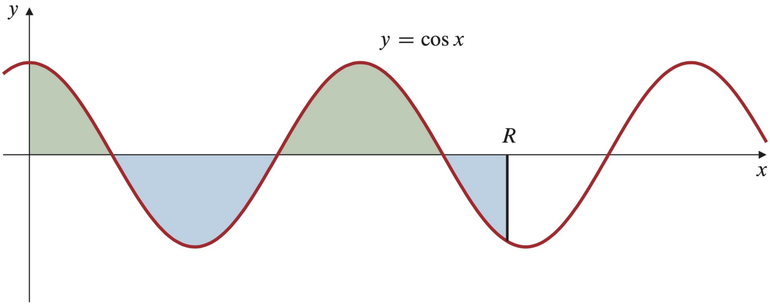


Improper Integrals

Improper Integrals of Type I

Example. $\int_0^\infty \cos x\,dx = \lim_{R\to\infty}\int_0^R \cos x\,dx = \lim_{R\to\infty} \sin R.$

This limit does not exist (and it is not ∞ or $-\infty$), so all we can say is that the given integral diverges.



Improper Integrals

Improper Integrals of Type II

Definition. *Improper Integrals of Type II*

If f is continuous on $(a, b]$ and is possibly unbounded near a, we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

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Similarly, if f is continuous on $[a, b)$ and is possibly unbounded near b, we define

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

These improper integrals may converge, diverge, diverge to infinity, or diverge to negative infinity.

Improper Integrals

Improper Integrals of Type II

Example. Find the area of the region S lying under $y = 1/\sqrt{x}$, above the x-axis, between $x = 0$ and $x = 1$.

Solution.

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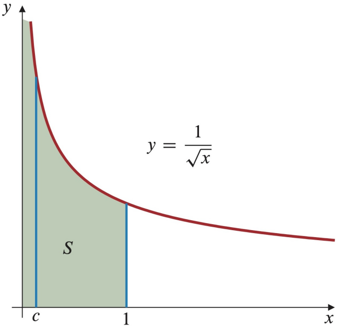
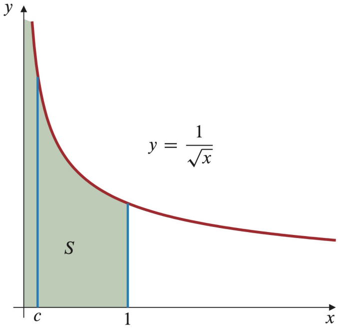


Improper Integrals

Improper Integrals of Type II

Example. Find the area of the region S lying under $y = 1/\sqrt{x}$, above the x-axis, between $x = 0$ and $x = 1$.

Solution. The area A is given by $A = \displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$, which is an improper integral of type II since the integrand is unbounded at $x = 0$.



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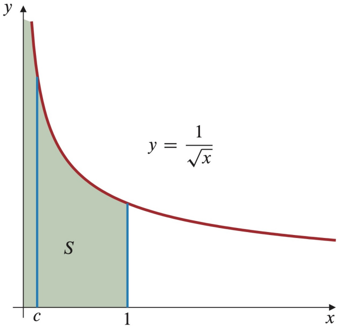

$$A = \lim_{c \to 0^+} \int_c^1 x^{-1/2}\,dx$$

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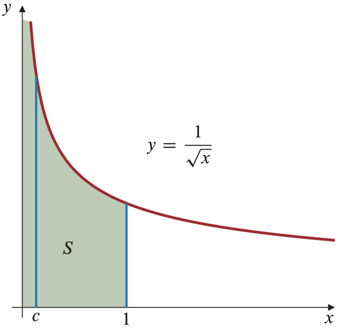


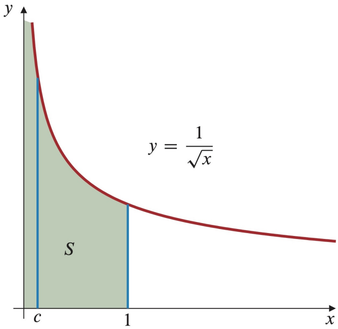
$$\begin{aligned} A &= \lim_{c\to 0^+} \int_c^1 x^{-1/2}\,dx \\ &= \lim_{c\to 0^+} 2x^{1/2}\Big|_c^1 \end{aligned}$$

Improper Integrals

Improper Integrals of Type II

Example. Find the area of the region S lying under $y = 1/\sqrt{x}$, above the x-axis, between $x = 0$ and $x = 1$.

Solution. The area A is given by $A = \int_0^1 \frac{1}{\sqrt{x}}\,dx$, which is an improper integral of type II since the integrand is unbounded at $x = 0$.



$$\begin{aligned} A &= \lim_{c\to 0^+} \int_c^1 x^{-1/2}\,dx \\ &= \lim_{c\to 0^+} 2x^{1/2}\Big|_c^1 \\ &= \lim_{c\to 0^+} (2 - 2\sqrt{c}) = 2. \end{aligned}$$

Improper Integrals

Improper Integrals of Type II

⚠

Be alert on the singularities!

Consider the integral

$$\int_{-1}^{1} \frac{\ln|x|\,dx}{\sqrt{1-x}}.$$

Improper Integrals

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This integral may seem like a definite integral, but it is improper at $x = 0$ and $x = 1$.

Improper Integrals

Improper Integrals of Type II

⚠

Be alert on the singularities!

Consider the integral

$$\int_{-1}^{1} \frac{\ln|x|\,dx}{\sqrt{1-x}}.$$

This integral may seem like a definite integral, but it is improper at $x = 0$ and $x = 1$.

$$\int_{-1}^{1} \frac{\ln|x|\,dx}{\sqrt{1-x}} = \int_{-1}^{0} \frac{\ln|x|\,dx}{\sqrt{1-x}} + \int_{0}^{1/2} \frac{\ln|x|\,dx}{\sqrt{1-x}} + \int_{1/2}^{1} \frac{\ln|x|\,dx}{\sqrt{1-x}}.$$

Each integral on the right is improper of type II because of a singularity at one endpoint.

Example. Evaluate each of the following integrals or show that it diverges:

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx, \quad \text{(b)}\ \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \quad \text{(c)}\ \int_0^1 \ln x\,dx.$$

Solution.

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx, \quad \text{(b)}\ \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \quad \text{(c)}\ \int_0^1 \ln x\,dx.$$

Solution.

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx$$

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

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Solution.

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x}\,dx$$

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx, \quad \text{(b)}\ \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \quad \text{(c)}\ \int_0^1 \ln x\,dx.$$

Solution.

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} \ln x\Big|_c^1$$

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx, \quad \text{(b)}\ \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \quad \text{(c)}\ \int_0^1 \ln x\,dx.$$

Solution.

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0^+}\int_c^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} \ln x\Big|_c^1 = \lim_{c\to 0^+}(0-\ln c) = \infty.$$

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

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Solution.

(b) $\displaystyle\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_0^2 \frac{1}{\sqrt{1-(x-1)^2}}\,dx$

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

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Solution.

(b) $\displaystyle\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_0^2 \frac{1}{\sqrt{1-(x-1)^2}}\,dx$

Let $u = x - 1$. Then $du = dx$.

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

$$\text{(a)} \int_0^1 \frac{1}{x}\,dx, \quad \text{(b)} \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \quad \text{(c)} \int_0^1 \ln x\,dx.$$

Solution.

$$\text{(b)} \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_0^2 \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$

Let $u = x - 1$. Then $du = dx$.

$$= \int_{-1}^1 \frac{1}{\sqrt{1-u^2}}\,du$$

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

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(Notice that the integrand is even!)

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

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Solution.

$$\text{(b)}\ \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_0^2 \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$

Let $u = x-1$. Then $du = dx$.

$$= \int_{-1}^1 \frac{1}{\sqrt{1-u^2}}\,du$$

(Notice that the integrand is even!)

$$= 2\int_0^1 \frac{1}{\sqrt{1-u^2}}\,du$$

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

$$\text{(a)} \int_0^1 \frac{1}{x}\,dx, \quad \text{(b)} \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \quad \text{(c)} \int_0^1 \ln x\,dx.$$

Solution.

$$\begin{aligned}
\text{(b)} \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx &= \int_0^2 \frac{1}{\sqrt{1-(x-1)^2}}\,dx \\
&= \int_{-1}^1 \frac{1}{\sqrt{1-u^2}}\,du \\
&= 2\int_0^1 \frac{1}{\sqrt{1-u^2}}\,du \\
&= 2\lim_{c\to 1^-}\int_0^c \frac{1}{\sqrt{1-u^2}}\,du
\end{aligned}$$

Let $u = x - 1$. Then $du = dx$.

(Notice that the integrand is even!)

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx, \quad \text{(b)}\ \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \quad \text{(c)}\ \int_0^1 \ln x\,dx.$$

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Let $u = x - 1$. Then $du = dx$.

$$= \int_{-1}^1 \frac{1}{\sqrt{1-u^2}}\,du$$

(Notice that the integrand is even!)

$$= 2\int_0^1 \frac{1}{\sqrt{1-u^2}}\,du$$

$$= 2\lim_{c\to 1^-}\int_0^c \frac{1}{\sqrt{1-u^2}}\,du$$

$$= 2\lim_{c\to 1^-}\left[\sin^{-1}(u)\right]_0^c = 2\lim_{c\to 1^-}\sin^{-1}(c) = \pi.$$

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx, \quad \text{(b)}\ \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \quad \text{(c)}\ \int_0^1 \ln x\,dx.$$

Solution.

$$\text{(c)}\ \int_0^1 \ln x\,dx = \lim_{c\to 0^+} \int_c^1 \ln x\,dx$$

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx, \quad \text{(b)}\ \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \quad \text{(c)}\ \int_0^1 \ln x\,dx.$$

Solution.

$$\text{(c)}\ \int_0^1 \ln x\,dx = \lim_{c\to 0^+} \int_c^1 \ln x\,dx = \lim_{c\to 0^+} \left[x\ln x - x\right]\Big|_c^1$$

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx, \quad \text{(b)}\ \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \quad \text{(c)}\ \int_0^1 \ln x\,dx.$$

Solution.

$$\begin{aligned}\text{(c)}\ \int_0^1 \ln x\,dx &= \lim_{c\to 0^+}\int_c^1 \ln x\,dx = \lim_{c\to 0^+}\left[x\ln x - x\right]\Big|_c^1 \\ &= \lim_{c\to 0^+}\left(0 - 1 - c\ln c + c\right)\end{aligned}$$

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx, \quad \text{(b)}\ \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \quad \text{(c)}\ \int_0^1 \ln x\,dx.$$

Solution.

$$\begin{aligned}
\text{(c)}\ \int_0^1 \ln x\,dx &= \lim_{c\to 0^+}\int_c^1 \ln x\,dx = \lim_{c\to 0^+}\left[x\ln x - x\right]\Big|_c^1 \\
&= \lim_{c\to 0^+}(0-1-c\ln c+c) \\
&= -1+0-\lim_{c\to 0^+}\frac{\ln c}{1/c} \quad \left[\frac{-\infty}{\infty}\right]
\end{aligned}$$

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx, \quad \text{(b)}\ \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \quad \text{(c)}\ \int_0^1 \ln x\,dx.$$

Solution.

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\text{(c)}\ \int_0^1 \ln x\,dx &= \lim_{c\to 0^+}\int_c^1 \ln x\,dx = \lim_{c\to 0^+}\left[x\ln x - x\right]\Big|_c^1 \\
&= \lim_{c\to 0^+}(0-1-c\ln c+c) \\
&= -1+0-\lim_{c\to 0^+}\frac{\ln c}{1/c} \quad \left[\frac{-\infty}{\infty}\right] \\
&= -1-\lim_{c\to 0^+}\frac{1/c}{-(1/c^2)} \quad \text{(by L'Hôpital's rule)}
\end{aligned}$$

Improper Integrals

Improper Integrals of Type II

Example. Evaluate each of the following integrals or show that it diverges:

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx, \quad \text{(b)}\ \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \quad \text{(c)}\ \int_0^1 \ln x\,dx.$$

Solution.

$$\begin{aligned}
\text{(c)}\ \int_0^1 \ln x\,dx &= \lim_{c\to 0^+} \int_c^1 \ln x\,dx = \lim_{c\to 0^+} \left[x\ln x - x\right]\Big|_c^1 \\
&= \lim_{c\to 0^+} (0 - 1 - c\ln c + c) \\
&= -1 + 0 - \lim_{c\to 0^+} \frac{\ln c}{1/c} \quad \left[\frac{-\infty}{\infty}\right] \\
&= -1 - \lim_{c\to 0^+} \frac{1/c}{-(1/c^2)} \quad \text{(by L'Hôpital's rule)} \\
&= -1 - \lim_{c\to 0^+} (-c) = -1.
\end{aligned}$$

Improper Integrals

Theorem. *p-integrals*

If $0 < a < \infty$, then

(a) $$\int_a^{\infty} \frac{1}{x^p}dx \quad \begin{cases} \text{converges to } \dfrac{a^{1-p}}{p-1} & \text{if } p > 1, \\ \text{diverges to } \infty & \text{if } p \leq 1. \end{cases}$$

Improper Integrals

Theorem. *p-integrals*

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(b) $$\int_0^a \frac{1}{x^p}dx \quad \begin{cases} \text{converges to } \dfrac{a^{1-p}}{1-p} & \text{if } p < 1, \\ \text{diverges to } \infty & \text{if } p \geq 1. \end{cases}$$

Improper Integrals

A Comparison Theorem for Integrals

Theorem. *A comparison theorem for integrals*

Let $-\infty \leq a < b \leq \infty$, and suppose that functions f and g are continuous on the interval (a, b) and satisfy $0 \leq f(x) \leq g(x)$. If $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ also converges, and

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Equivalently, if $\int_a^b f(x)dx$ diverges to ∞, then $\int_a^b g(x)dx$ also diverges to ∞.

Improper Integrals

Example. Show that $\int_0^\infty e^{-x^2}\,dx$ converges, and find an upper bound for its value.

Solution. On $[0,1]$, we have $0 < e^{-x^2} \leq 1$.

Improper Integrals

Example. Show that $\int_0^\infty e^{-x^2}dx$ converges, and find an upper bound for its value.

Solution. On $[0,1]$, we have $0 < e^{-x^2} \leq 1$. So

$$0 < \int_0^1 e^{-x^2}dx \leq \int_0^1 dx = 1.$$

Improper Integrals

Example. Show that $\int_0^\infty e^{-x^2} dx$ converges, and find an upper bound for its value.

Solution. On $[0, 1]$, we have $0 < e^{-x^2} \leq 1$. So

$$0 < \int_0^1 e^{-x^2} dx \leq \int_0^1 dx = 1.$$

On $[1, \infty)$, we have $x^2 \geq x$, so $-x^2 \leq -x$ and $0 < e^{-x^2} \leq e^{-x}$.

Improper Integrals

Example. Show that $\int_0^\infty e^{-x^2}\,dx$ converges, and find an upper bound for its value.

Solution. On $[0,1]$, we have $0 < e^{-x^2} \leq 1$. So

$$0 < \int_0^1 e^{-x^2}\,dx \leq \int_0^1 dx = 1.$$

On $[1,\infty)$, we have $x^2 \geq x$, so $-x^2 \leq -x$ and $0 < e^{-x^2} \leq e^{-x}$. Thus,

$$\begin{aligned} 0 < \int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x}\,dx &= \lim_{R\to\infty} \frac{e^{-x}}{-1}\bigg|_1^R \\ &= \lim_{R\to\infty}\left(\frac{1}{e} - \frac{1}{e^R}\right) = \frac{1}{e}. \end{aligned}$$

Improper Integrals

Example. Show that $\int_0^\infty e^{-x^2}dx$ converges, and find an upper bound for its value.

Solution. On $[0,1]$, we have $0 < e^{-x^2} \le 1$. So

$$0 < \int_0^1 e^{-x^2}dx \le \int_0^1 dx = 1.$$

On $[1,\infty)$, we have $x^2 \ge x$, so $-x^2 \le -x$ and $0 < e^{-x^2} \le e^{-x}$. Thus,

$$\begin{aligned} 0 < \int_1^\infty e^{-x^2}dx \le \int_1^\infty e^{-x}dx &= \lim_{R\to\infty} \frac{e^{-x}}{-1}\bigg|_1^R \\ &= \lim_{R\to\infty} \left(\frac{1}{e} - \frac{1}{e^R}\right) = \frac{1}{e}. \end{aligned}$$

Therefore, $\int_0^\infty e^{-x^2}dx$ converges, and its value is positve and less than or equal to $1 + \frac{1}{e}$.