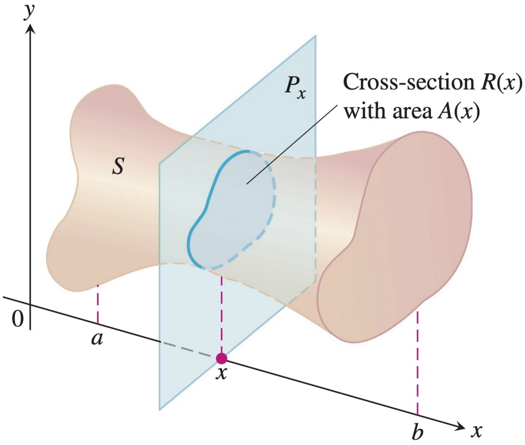
MAT123 MATHEMATICS I

Lecture 21: Applications of Integration

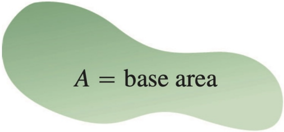
Outline

Volumes By Slicing

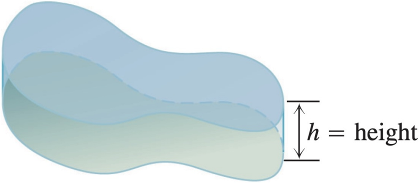
Volumes By Slicing



Volumes By Slicing

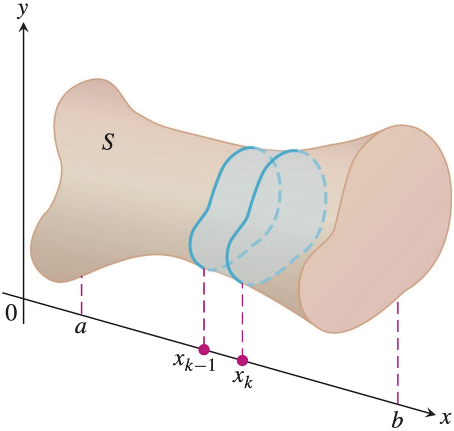


Plane region whose
area we know



Cylindrical solid based on region
Volume = base area × height = Ah

Volumes By Slicing



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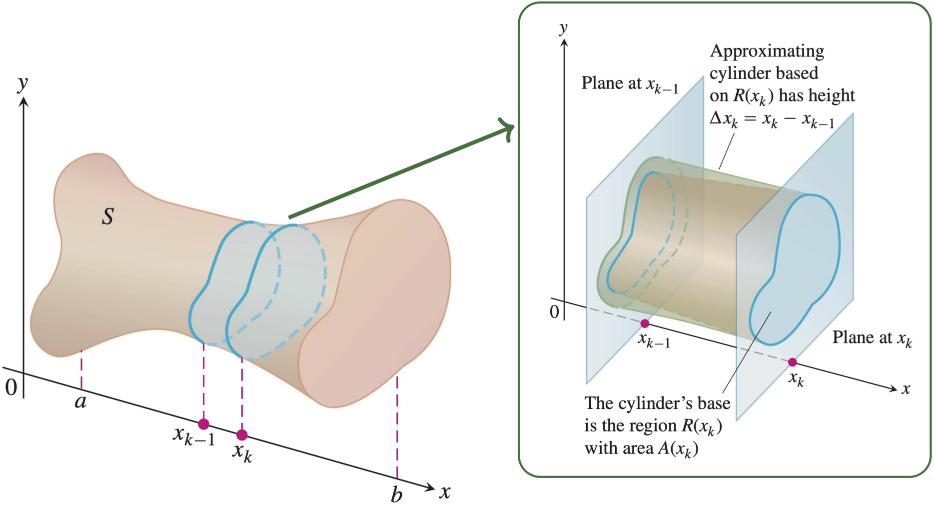

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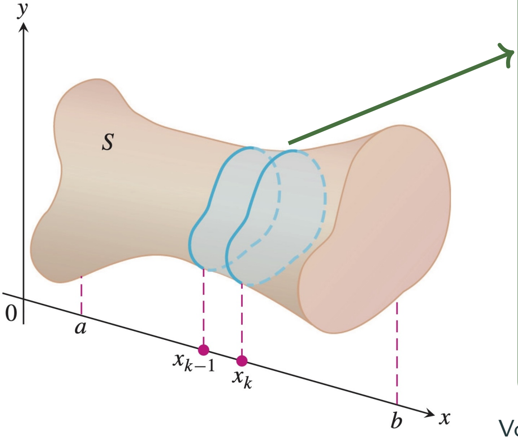
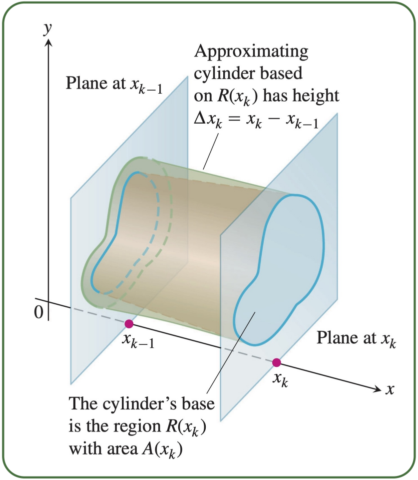
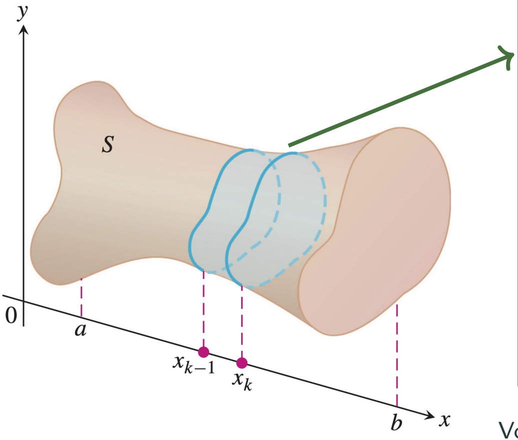
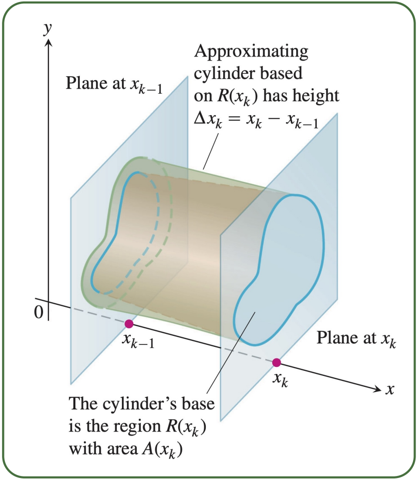




Volume of the kth slab $\approx V_k = A(x_k)\,\Delta x_k$

Volumes By Slicing





Plane at x_{k-1}

Approximating cylinder based on $R(x_k)$ has height $\Delta x_k = x_k - x_{k-1}$

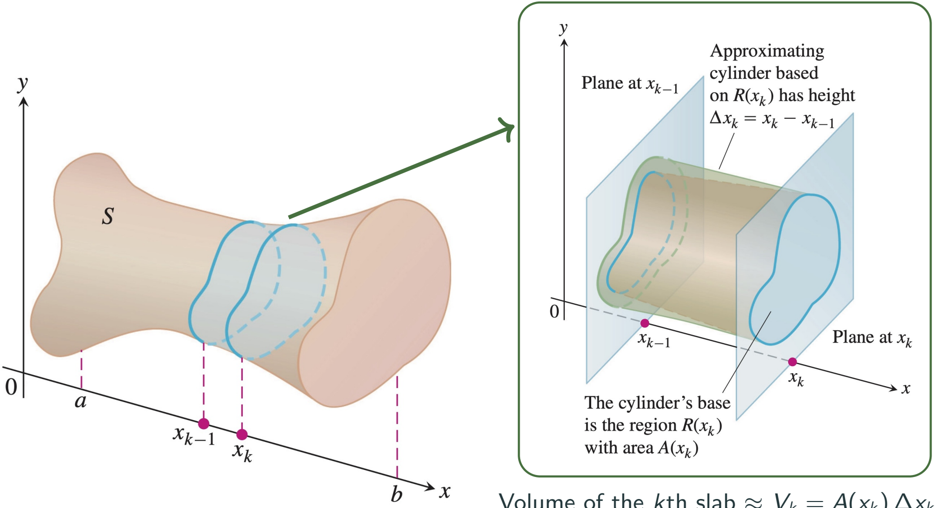
Plane at x_k

The cylinder's base is the region $R(x_k)$ with area $A(x_k)$

Volume of the kth slab $\approx V_k = A(x_k)\,\Delta x_k$

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\,\Delta x_k$$

Volumes By Slicing



Volume of the kth slab $\approx V_k = A(x_k)\,\Delta x_k$

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\,\Delta x_k \quad \Rightarrow \quad V = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\,\Delta x_k = \int_a^b A(x)\,dx$$

Volumes By Slicing

Definition.

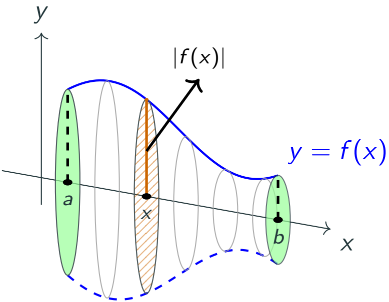
The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

Volumes By Slicing

Solids of Revolution: The Disk Method

If the region R bounded by $y = f(x), y = 0, x = a$, and $x = b$ is rotated about the x-axis, then the cross-section of the solid generated in the plane perpendicular to the x-axis at x is a circular disk of radius $|f(x)|$.



Volumes By Slicing

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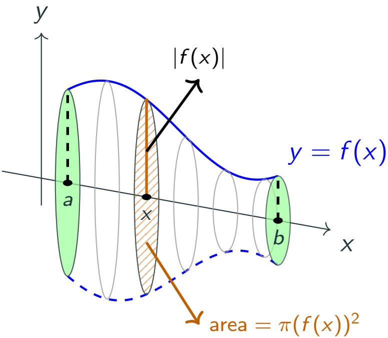
$$V = \pi \int_a^b (f(x))^2 \, dx.$$



Volumes By Slicing – Solids of Revolution

Solids of Revolution: The Disk Method

Example. **(The volume of a ball)** Find the volume of a solid ball having radius a.

Solution.

Volumes By Slicing – Solids of Revolution

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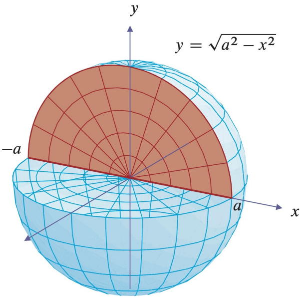

The ball is a solid of revolution generated by rotating the semicircle $y = \sqrt{a^2 - x^2}$ about the x-axis.

Volumes By Slicing – Solids of Revolution

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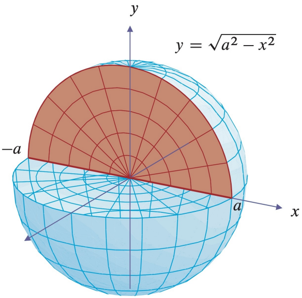

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Volumes By Slicing – Solids of Revolution

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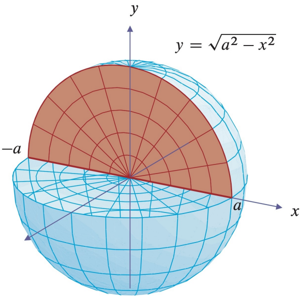

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Volumes By Slicing – Solids of Revolution

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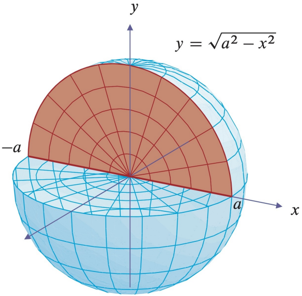

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$$= 2\pi \left(a^2 x - \frac{x^3}{3} \right) \Big|_0^a$$

Volumes By Slicing – Solids of Revolution

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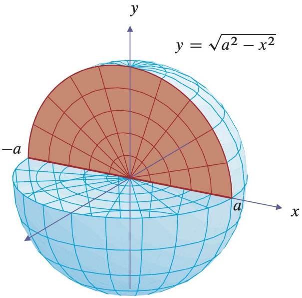

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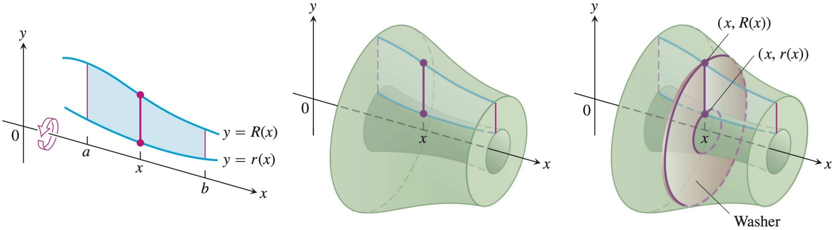
$$V = \pi \int_{-a}^{a} (\sqrt{a^2 - x^2})^2\, dx = 2\pi \int_0^a (a^2 - x^2) dx$$

$$= 2\pi \left(a^2 x - \frac{x^3}{3} \right) \Bigg|_0^a = 2\pi \left(a^3 - \frac{a^3}{3} \right)$$

$$= \frac{4}{3} \pi a^3.$$

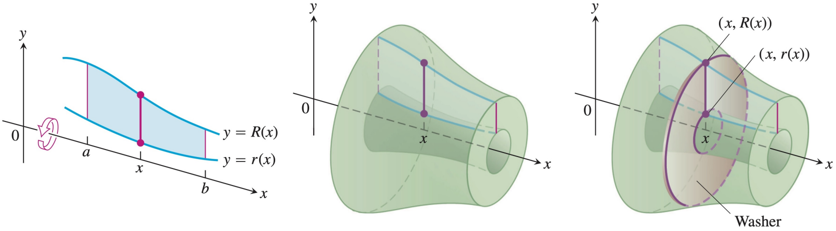
Volumes By Slicing

Solids of Revolution: The Washer Method



Volumes By Slicing

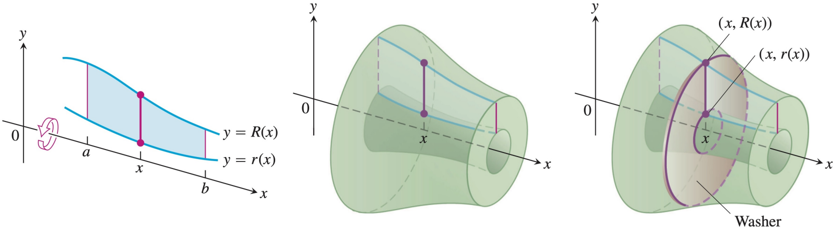
Solids of Revolution: The Washer Method



The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)dx$ leads to a slightly different formula.

Volumes By Slicing

Solids of Revolution: The Washer Method



The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)dx$ leads to a slightly different formula.

$$V = \int_a^b A(x)dx = \int_a^b \pi \left([R(x)]^2 - [r(x)]^2\right) dx.$$

Volumes By Slicing – Solids of Revolution

Solids of Revolution

Example. A ring-shaped solid is generated by rotating the finite plane region R bounded by the curve $y = x^2$ and the line $y = 1$ about the line $y = 2$. Find its volume.

Volumes By Slicing – Solids of Revolution

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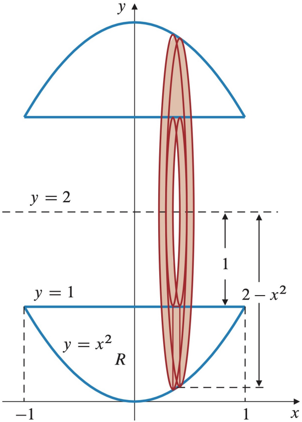

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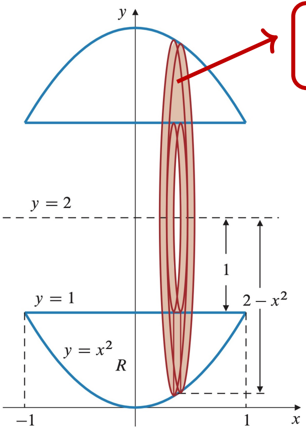

$$\text{cross-sectional area} \quad = \pi(2 - x^2)^2 - \pi(1)^2$$
$$= \pi(3 - 4x^2 + x^4)$$

Volumes By Slicing – Solids of Revolution

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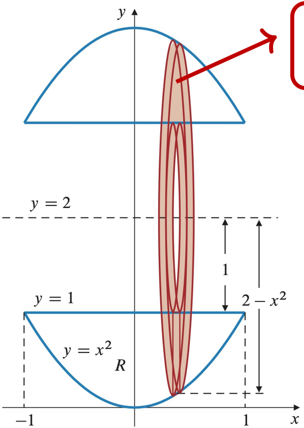

$$\begin{aligned} \text{cross-sectional area} \quad &= \pi(2 - x^2)^2 - \pi(1)^2 \\ &= \pi(3 - 4x^2 + x^4) \end{aligned}$$

$$V \quad = \pi \int_{-1}^{1} (3 - 4x^2 + x^4)\,dx$$

Volumes By Slicing – Solids of Revolution

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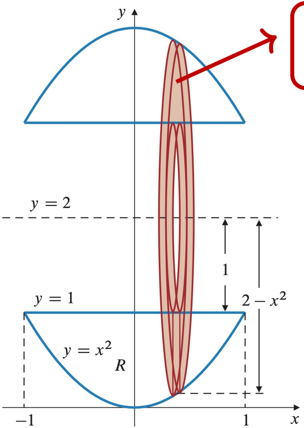

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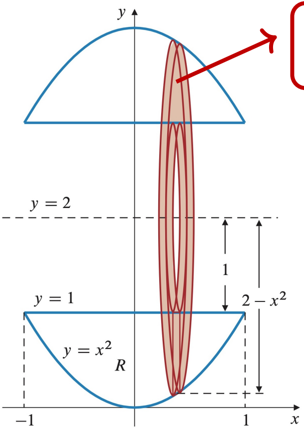

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Volumes By Slicing – Solids of Revolution

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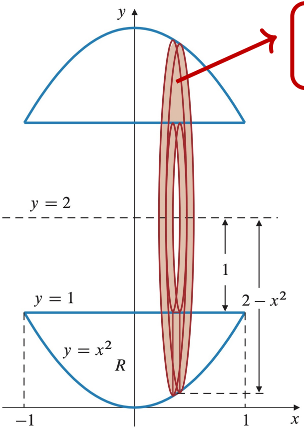

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Volumes By Slicing

Solids of Revolution

Example. Find the volume of the solid generated by rotating the region to the right of the y-axis and to the left of the curve $x = 2y - y^2$ about the y-axis.

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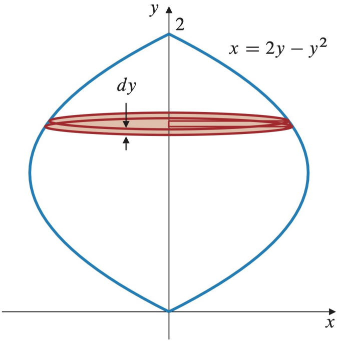

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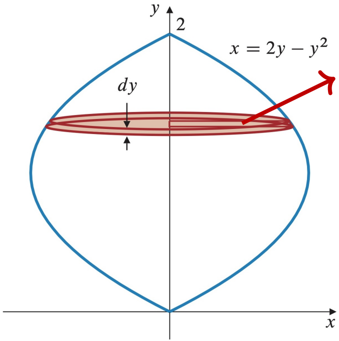

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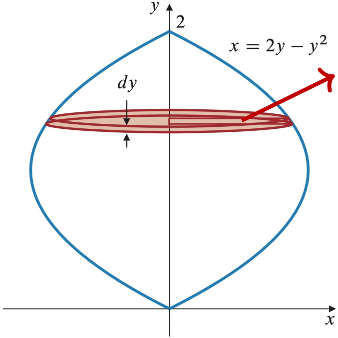

$$\begin{aligned}\text{cross-sectional area} &= \pi(2y - y^2)^2 \\ &= \pi(4y^2 - 4y^3 + y^4)\end{aligned}$$

$$\begin{aligned} V &= \pi \int_0^2 (4y^2 - 4y^3 + y^4) dy \\ &= \pi \left(\frac{4y^3}{3} - y^4 + \frac{y^5}{5} \right) \Big|_0^2 \\ &= \pi \left(\frac{32}{3} - 16 + \frac{32}{5} \right) \\ &= \frac{16\pi}{15}. \end{aligned}$$

Volumes By Slicing

Solids of Revolution: Cylindrical Shells

Some volumes are difficult to compute using the washer method.

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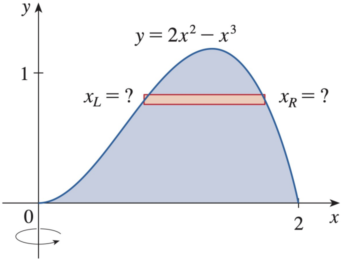

- **Example:** rotate the region between

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

about the y-axis.

Volumes By Slicing

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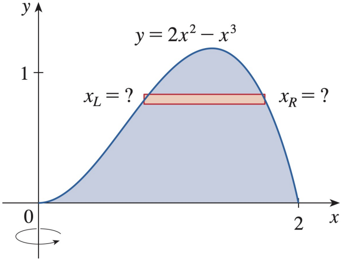

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- Slicing perpendicular to the y-axis produces washers.

Volumes By Slicing

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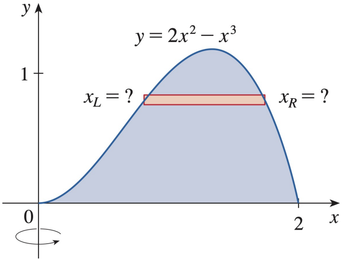

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- But the inner and outer radii require solving

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  for x in terms of y.

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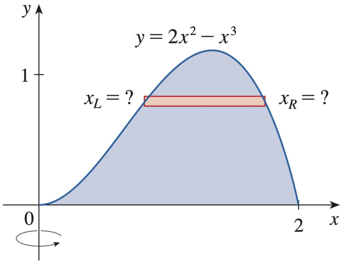

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- This is a cubic equation, so inversion is not easy.

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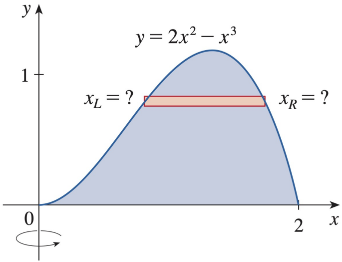

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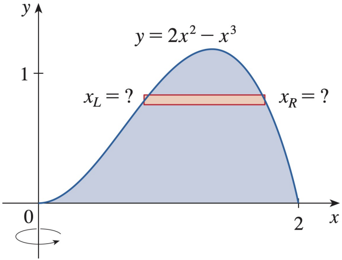

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- But the inner and outer radii require solving

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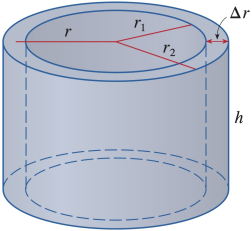
We use a different method (called **cylindrical shells**) to compute such volumes!

Volumes By Slicing

Solids of Revolution: Cylindrical Shells



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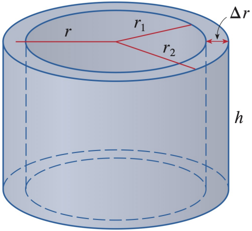
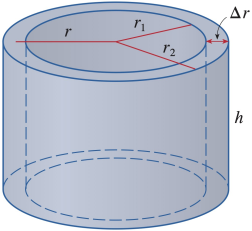
Solids of Revolution: Cylindrical Shells



The figure shows a cylindrical shell (a can without a top or bottom) with inner radius r_1, outer radius r_2, and height h.

Volumes By Slicing

Solids of Revolution: Cylindrical Shells

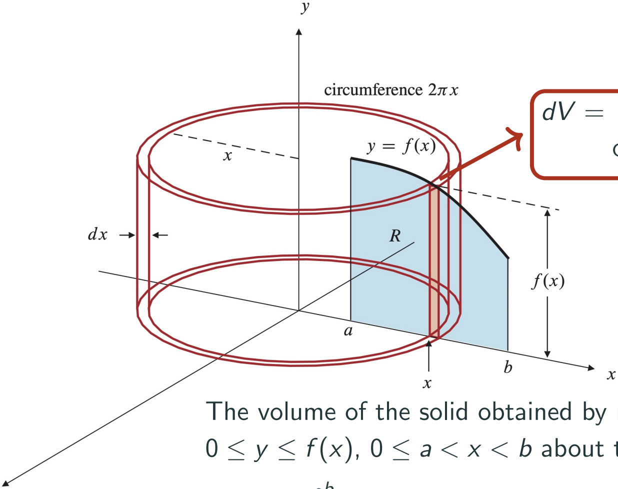


The figure shows a cylindrical shell (a can without a top or bottom) with inner radius r_1, outer radius r_2, and height h. The volume of this shell is

$$\begin{aligned} V &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 - r_1)(r_2 + r_1)h \\ &= 2\pi\left(\frac{r_1 + r_2}{2}\right)(r_2 - r_1)h \\ &= 2\pi(\text{average radius})(\text{thickness})(\text{height}) = 2\pi r\, h\, \Delta_r. \end{aligned}$$

Volumes By Slicing

Solids of Revolution: Cylindrical Shells



$$\boxed{dV = \underbrace{2\pi x}_{\text{Circumference}} \cdot \underbrace{f(x)}_{\text{Height}} \cdot \underbrace{dx}_{\text{Thickness}}}$$

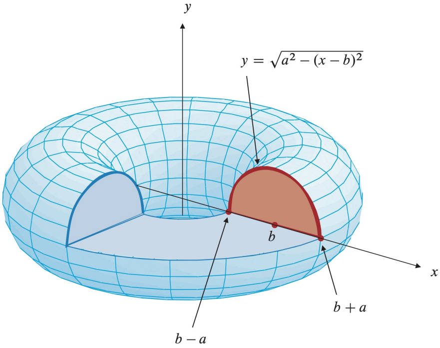
The volume of the solid obtained by rotating the plane region $0 \leq y \leq f(x)$, $0 \leq a < x < b$ about the y-axis is

$$V = 2\pi \int_a^b x\, f(x)\, dx\,; \quad x : \text{shell radius} \quad f(x) : \text{shell height}$$

Volumes By Slicing

Solids of Revolution

Example. **(The volume of a torus)** A disk of radius a has centre at the point $(b, 0)$, where $b > a > 0$. The disk is rotated about the y-axis to generate a **torus** (a doughnut-shaped solid). Find its volume.



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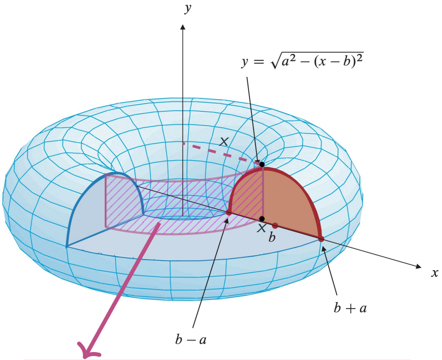

shell at x $(b - a < x < b + a)$
has radius x and height $\sqrt{a^2 - (x-b)^2}$

Volumes By Slicing

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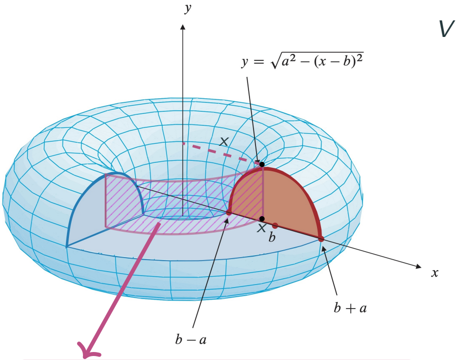

$$V = 2 \times 2\pi \int_{b-a}^{b+a} x\sqrt{a^2 - (x-b)^2}\, dx$$

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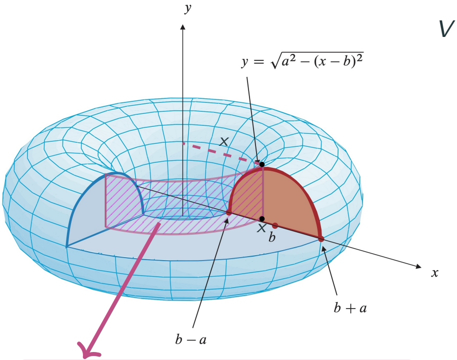

$$V = 2 \times 2\pi \int_{b-a}^{b+a} x\sqrt{a^2 - (x-b)^2}\, dx$$

Let $u = x - b$. Then $du = dx$.

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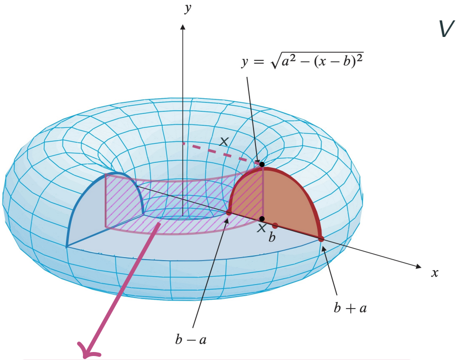

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Let $u = x - b$. Then $du = dx$.

$$= 4\pi \int_{-a}^{a} (u + b)\sqrt{a^2 - u^2}\,du$$

Volumes By Slicing

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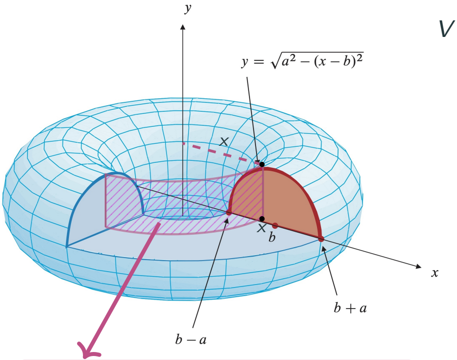

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Let $u = x - b$. Then $du = dx$.

$$= 4\pi \int_{-a}^{a} (u + b)\sqrt{a^2 - u^2}\, du$$

$$= 4\pi \left(\int_{-a}^{a} u\sqrt{a^2 - u^2}\, du + b \int_{-a}^{a} \sqrt{a^2 - u^2}\, du \right)$$

Volumes By Slicing

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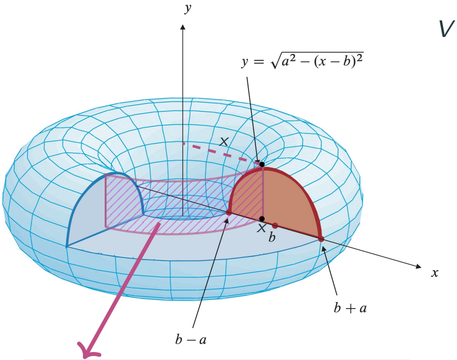

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Let $u = x - b$. Then $du = dx$.

$$= 4\pi \int_{-a}^{a} (u+b)\sqrt{a^2 - u^2}\,du$$

$$= 4\pi\left(\int_{-a}^{a} u\sqrt{a^2 - u^2}\,du \right.$$

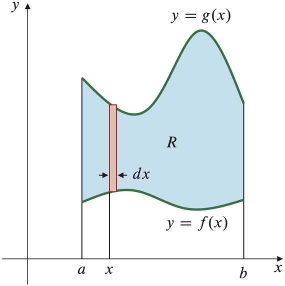
$$\left. +b\int_{-a}^{a} \sqrt{a^2 - u^2}\,du\right)$$

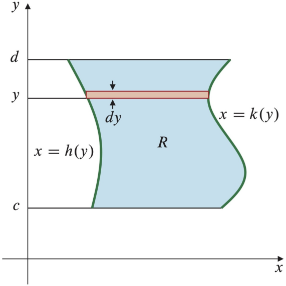
$$= 0 + 4\pi b\frac{\pi a^2}{2} = 2\pi^2 a^2 b.$$

Volumes By Slicing

Solids of Revolution

If region $R \longrightarrow$





is rotated about

$\downarrow$

the x-axis	use plane slices $V = \pi \int_a^b \left((g(x))^2 - (f(x))^2 \right) dx$	use cylindrical shells $V = 2\pi \int_c^d y \left(k(y) - h(y) \right) dy$
the y-axis	use cylindrical shells $V = 2\pi \int_a^b x \left(g(x) - f(x) \right) dx$	use plane slices $V = \pi \int_c^d \left((k(y))^2 - (h(y))^2 \right) dy$

Volumes By Slicing

Solids of Revolution

Shell Formula for Revolution About a Vertical Line

$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx$$

Volumes By Slicing

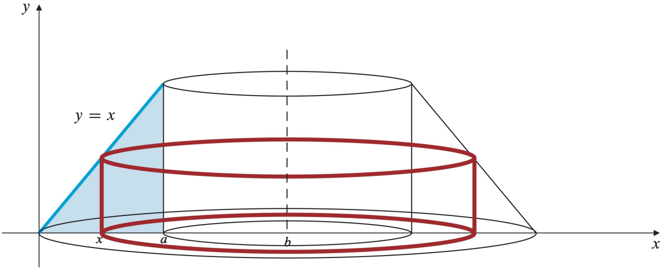
Solids of Revolution

Example. The triangular region bounded by $y = x$, $y = 0$, and $x = a > 0$ is rotated about the line $x = b > a$. Find the volume of the solid generated.

Volumes By Slicing

Solids of Revolution

Example. The triangular region bounded by $y = x$, $y = 0$, and $x = a > 0$ is rotated about the line $x = b > a$. Find the volume of the solid generated.



Volumes By Slicing

Solids of Revolution

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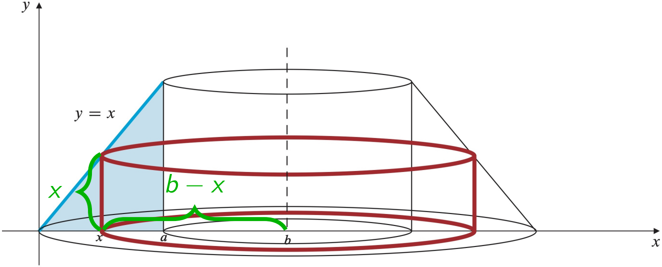

Volumes By Slicing

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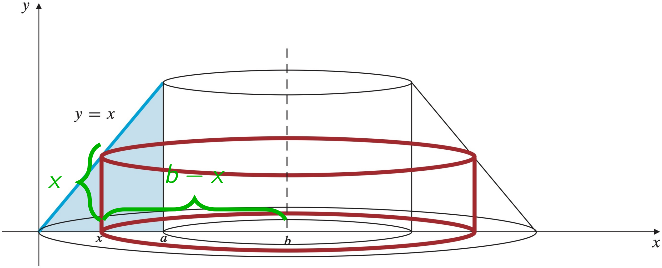

$$V = 2\pi \int_0^a (b - x)x \, dx$$

Volumes By Slicing

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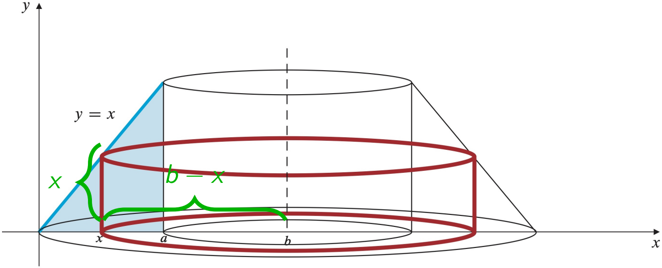
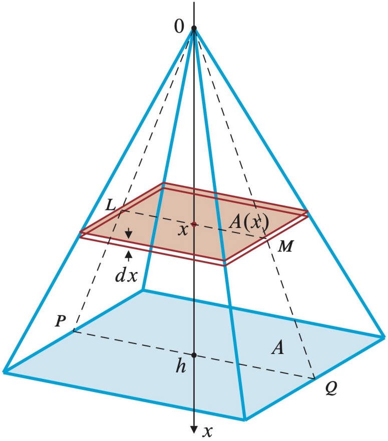


$$V = 2\pi \int_0^a (b - x)x\,dx = 2\pi \left(\frac{bx^2}{2} - \frac{x^3}{3} \right) \Big|_0^a = \pi \left(a^2 b - \frac{2a^3}{3} \right)$$

More Volumes By Slicing

Example. Verify the formula for the volume of a pyramid with rectangular base of area A and height h.



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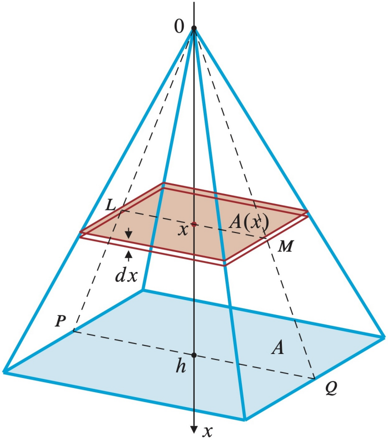

From similar triangles OLM and OPQ, the area of the rectangular cross-section at x is

$$A(x) = \left(\frac{x}{h}\right)^2 A.$$

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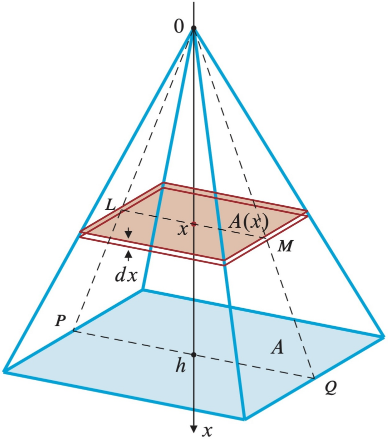

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$$A(x) = \left(\frac{x}{h}\right)^2 A.$$

Thus, the volume of the pyramid is

$$V = \int_0^h \left(\frac{x}{h}\right)^2 A\,dx = \frac{A}{h^2}\frac{x^3}{3}\bigg|_0^h$$

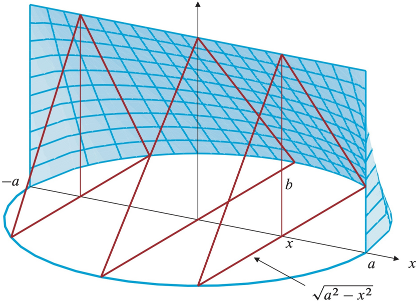
$$= \frac{1}{3}Ah.$$

More Volumes By Slicing

Example. A tent has a circular base of radius a metres and is supported by a horizontal ridge bar held at height b metres above a diameter of the base by vertical supports at each end of the diameter. The material of the tent is stretched tight so that each cross-section perpendicular to the ridge bar is an isosceles triangle. Find the volume of the tent.

More Volumes By Slicing

Example. A tent has a circular base of radius *a* metres and is supported by a horizontal ridge bar held at height *b* metres above a diameter of the base by vertical supports at each end of the diameter. The material of the tent is stretched tight so that each cross-section perpendicular to the ridge bar is an isosceles triangle. Find the volume of the tent.



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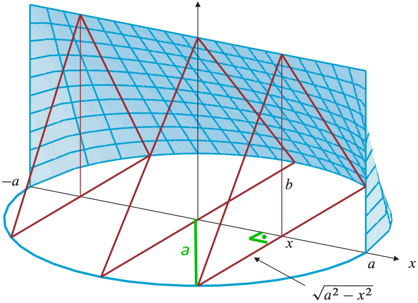

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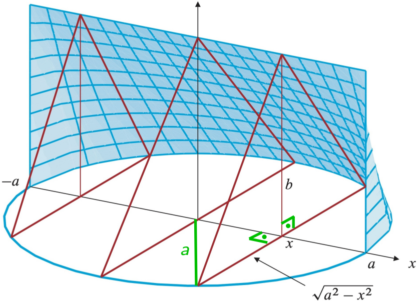

$$\boxed{A(x) = \frac{1}{2}(2\sqrt{a^2-x^2})b = b\sqrt{a^2-x^2}}$$

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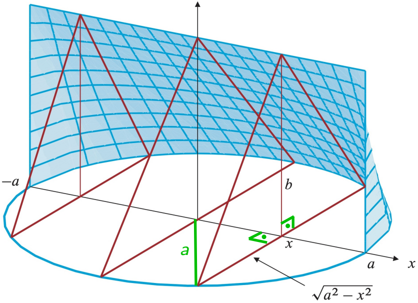


$$A(x) = \frac{1}{2}(2\sqrt{a^2-x^2})b = b\sqrt{a^2-x^2}$$

Thus, the volume of the tent is

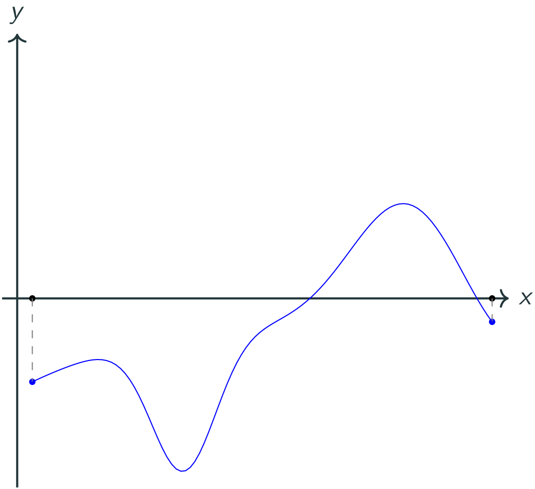
$$V = \int_{-a}^{a} b\sqrt{a^2-x^2}dx = b\int_{-a}^{a}\sqrt{a^2-x^2}dx$$

$$= b\frac{\pi a^2}{2} = \frac{\pi}{2}a^2 b.$$

Arc Length and Surface Area

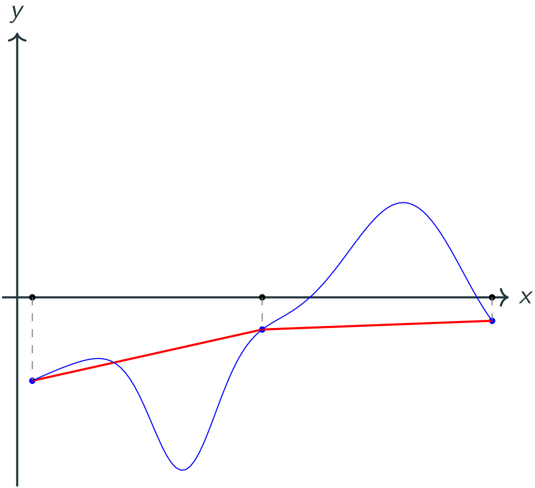
Arc Length and Surface Area

How can we find the length of the following curve?

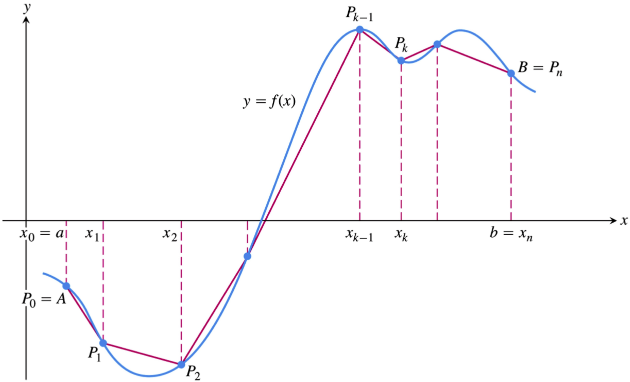


Arc Length and Surface Area

Subdivide the curve into many pieces and join successive points of division by straight line segments. Adding more points produces a closer approximation of the curve.



Arc Length and Surface Area



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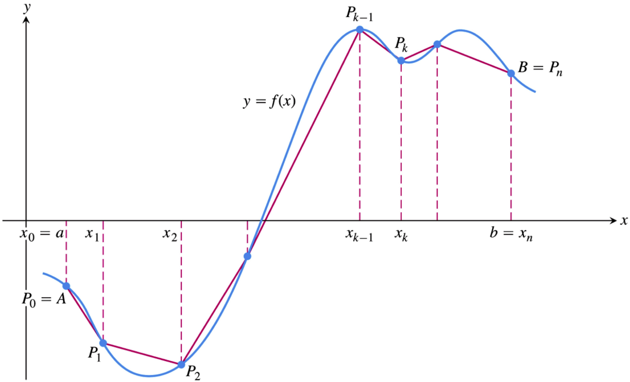

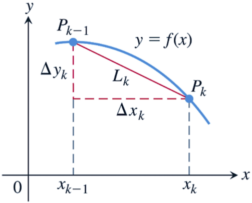

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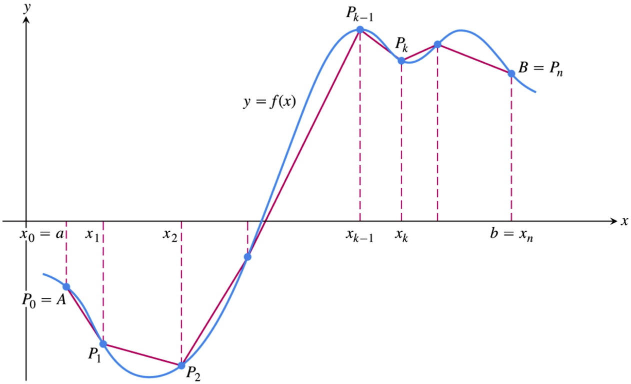
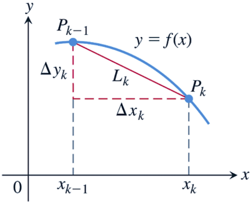
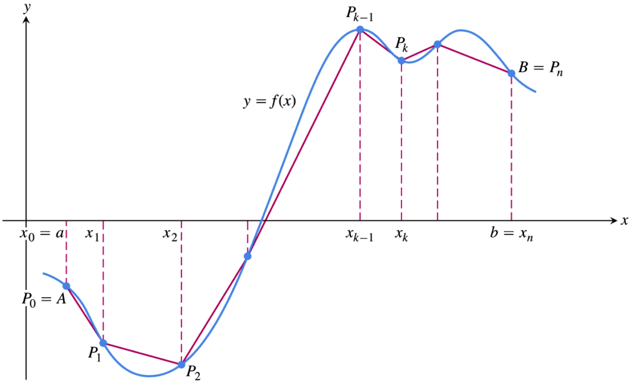


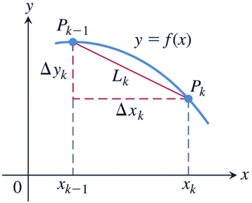


Now, by the Mean Value Theorem, there is a point c_k, with $x_{k-1} < c_k < x_k$, such that

$$\Delta y_k = f'(c_k)\Delta x_k.$$

Arc Length and Surface Area





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$$\Delta y_k = f'(c_k)\Delta x_k.$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} L_k = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

Arc Length and Surface Area

Definition.

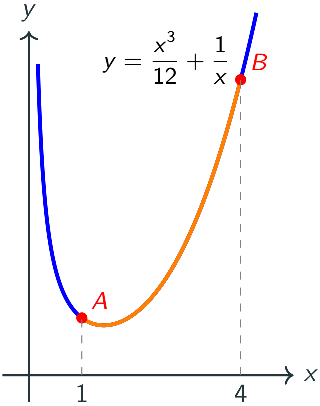
If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Arc Length and Surface Area

Example. Find the length of the graph of $f(x) = \dfrac{x^3}{12} + \dfrac{1}{x}$, $1 \le x \le 4$.

Solution.



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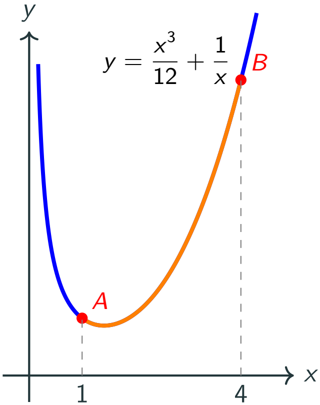

- $f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$

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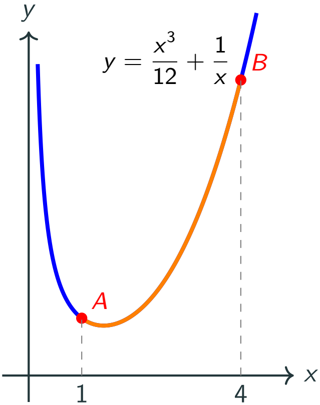

- $f'(x) = \dfrac{x^2}{4} - \dfrac{1}{x^2}$

- $$1 + [f'(x)]^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right)$$
  $$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2.$$

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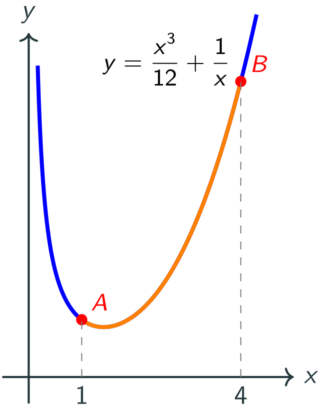

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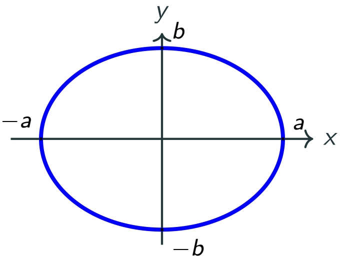
$$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2.$$

The length of the graph over $[1,4]$ is

$$L = \int_1^4 \sqrt{1 + [f'(x)]^2}\, dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$$
$$= \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4 = \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right) = 6.$$

Arc Length and Surface Area

Example. **(The circumference of an ellipse)** Find the circumference of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a \geq b > 0$.



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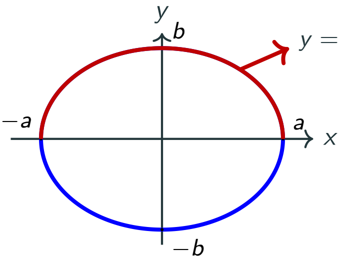

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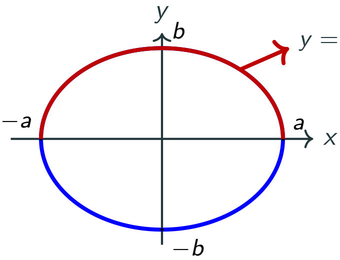

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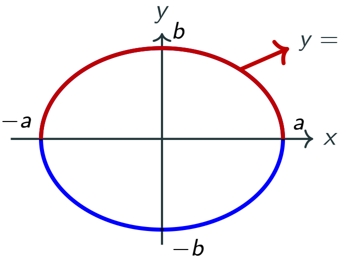

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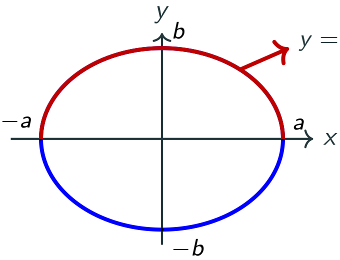

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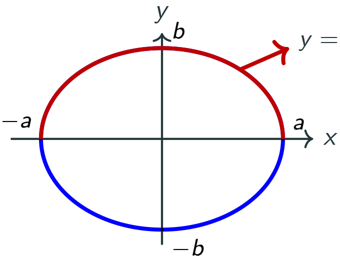

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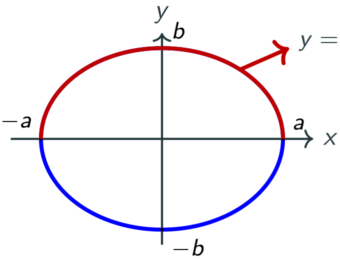

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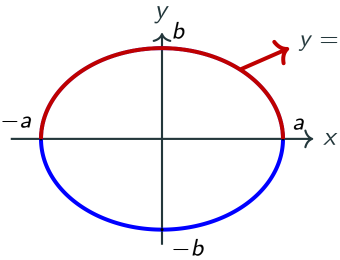

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Arc Length and Surface Area

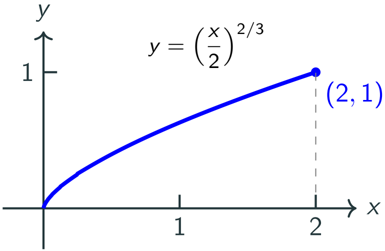
Formula for the Length of $x = g(y)$, $c \leq y \leq d$.

If g' is continuous on $[c, d]$, then the length of the curve $x = g(y)$ from the point $A = (g(c), c)$ to the point $B = (g(d), d)$ is the value of the integral

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Arc Length and Surface Area

Example. Find the length of the graph of $f(x) = (x/2)^{2/3}$, $0 \le x \le 2$.



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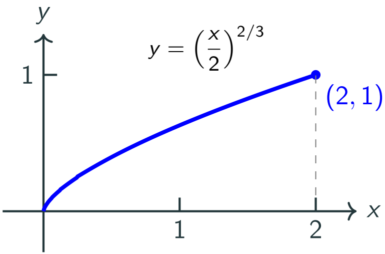

The derivative

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

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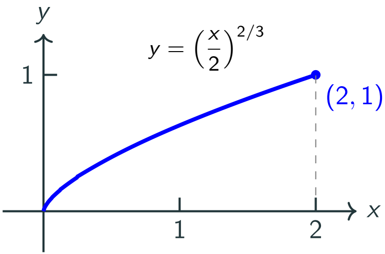

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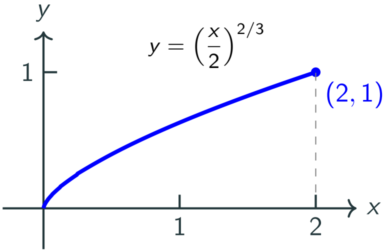

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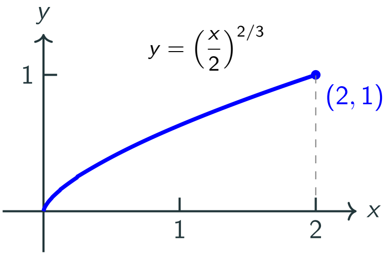

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$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_0^1 \sqrt{1+9y}\,dy$$

$$= \frac{1}{9}\frac{2}{3}(1+9y)^{3/2}\Big|_0^1 = \frac{2}{27}(10\sqrt{10}-1) \approx 2.27.$$

Arc Length and Surface Area

The Differential Formula for Arc Length

If $y = f(x)$ and if f' is continuous on $[a, b]$, then by the Fundamental Theorem of Calculus we can define a new function

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

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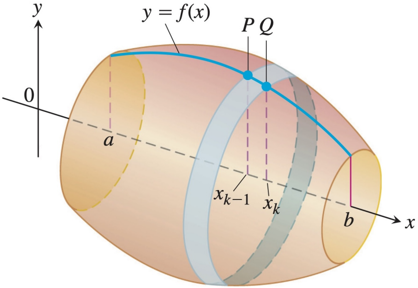
Then the differential of arc length is

$$\begin{aligned} ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \sqrt{dx^2 + dy^2}. \end{aligned}$$

Arc Length and Surface Area

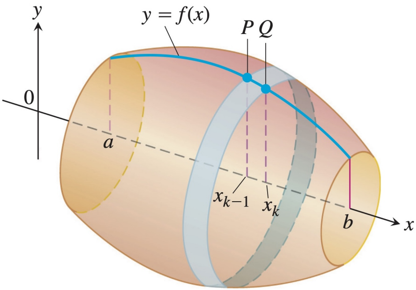
Areas of Surfaces of Revolution

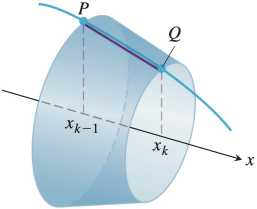


The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \leq x \leq b$, about the x-axis. The surface is a union of bands like the one swept out by the arc PQ.

Arc Length and Surface Area

Areas of Surfaces of Revolution





The line segment joining P and Q sweeps out a frustum of a cone.

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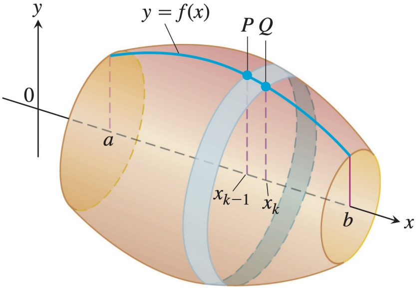

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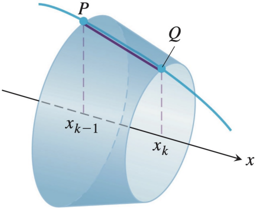

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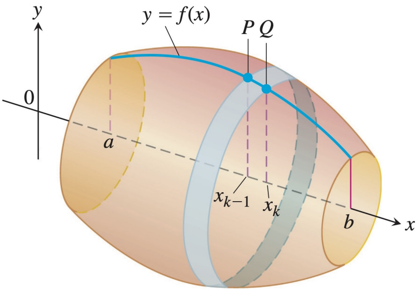
Frustum surface area:

$$= 2\pi \frac{f(x_{k-1}) + f(x_k)}{2} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$= \pi \left(f(x_{k-1}) + f(x_k)\right) \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

Arc Length and Surface Area

Areas of Surfaces of Revolution



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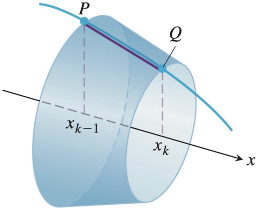

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Arc Length and Surface Area

Areas of Surfaces of Revolution

Example. **(Surface Area of a Sphere)** Find the area of the surface of a sphere of radius a.

Arc Length and Surface Area

Areas of Surfaces of Revolution

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Solution. Such a sphere can be generated by rotating the semicircle with equation $y = \sqrt{a^2 - x^2}$, $(-a \leq x \leq a)$, about the x-axis.

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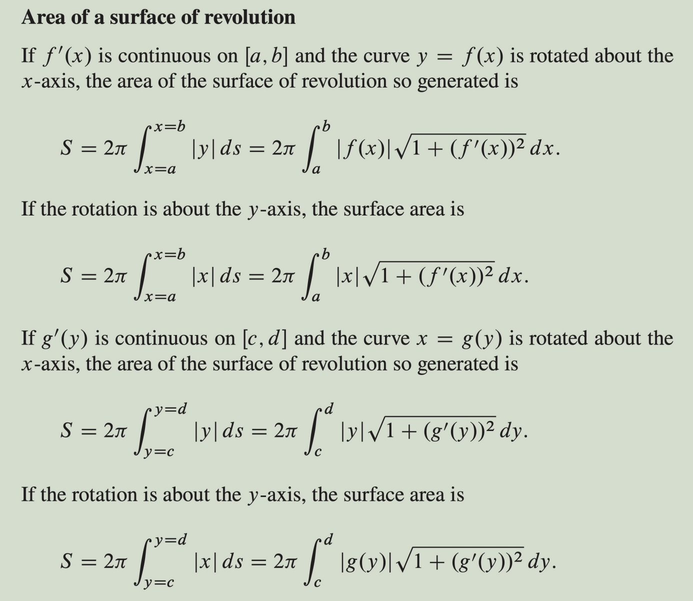
Arc Length and Surface Area

Areas of Surfaces of Revolution



Area of a surface of revolution

If $f'(x)$ is continuous on $[a, b]$ and the curve $y = f(x)$ is rotated about the x-axis, the area of the surface of revolution so generated is

$$S = 2\pi \int_{x=a}^{x=b} |y|\, ds = 2\pi \int_a^b |f(x)| \sqrt{1 + (f'(x))^2}\, dx.$$

If the rotation is about the y-axis, the surface area is

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Example. Find the surface area of a parabolic reflector whose shape is obtained by rotating the parabolic arc $y = x^2$, $0 \leq x \leq 1$, about the y-axis.

Solution.

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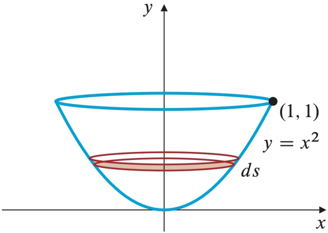

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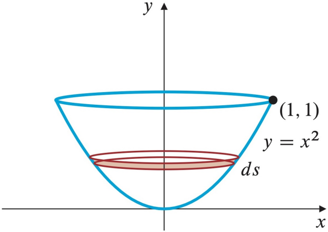

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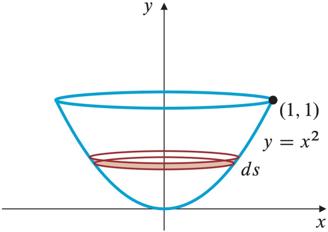

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$$S = 2\pi \int_0^1 x\sqrt{1 + 4x^2}\, dx$$

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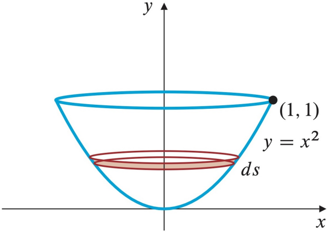

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$$S = 2\pi \int_0^1 x\sqrt{1 + 4x^2}\,dx$$

Let $u = 1 + 4x^2$. Then $du = 8x\,dx$.

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Areas of Surfaces of Revolution

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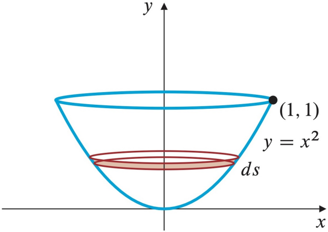

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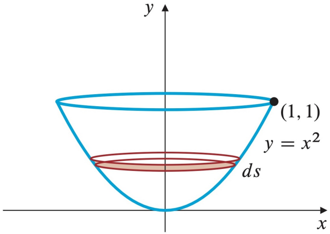

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