

# **MAT123 MATHEMATICS I**

## Lecture 22: Infinite Sequences and Series

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## Sequences

Representing Sequences

Convergence and Divergence

Calculating Limits of Sequences

Commonly Occurring Limits

# Sequences

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# Sequences

## Representing Sequences

A sequence is a list of numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

in a given order. So each of  $a_1, a_2, a_3$  and so on represents a number. These are the **terms** of the sequence.

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has first term  $a_1 = 2$ , second term  $a_2 = 4$ , and  $n$ th term  $a_n = 2n$ .

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Order is important. The sequence  $2, 4, 6, 8, \dots$  is not the same as the sequence  $4, 2, 6, 8, \dots$

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We can think of the sequence

$$a_1, a_2, a_3, \dots, a_n, \dots$$

as a function that sends 1 to  $a_1$ , 2 to  $a_2$ , and in general sends the positive integer  $n$  to the  $n$ th term  $a_n$ . More precisely, an **infinite sequence** of numbers is a function whose domain is the set of positive integers.

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$$2, 4, 6, 8, 10, \dots, 2n, \dots$$

sends 1 to  $a_1 = 2$ , 2 to  $a_2 = 4$ , and so on. The general behavior of this sequence is described by the formula  $a_n = 2n$  (or the function  $f(n) = 2n$ ).

# Sequences

## Representing Sequences

We can equally well make the domain the integers larger than a given number  $n_0$ , and we allow sequences of this type also. For example, the sequence

$$12, 14, 16, 18, 20, \dots$$

is described by the formula  $a_n = 10 + 2n$ . It can also be described by the simpler formula  $b_n = 2n$ , where the index  $n$  starts at 6 and increases. To allow such simpler formulas, we let the first index of the sequence be any integer.

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- (iii) By giving a recursive formula, which gives the first term and a rule for finding each term from the previous one(s).

The terms of a sequence are usually listed in braces, for example

$$\{a_n\} = \{2, 4, 6, 8, 10, \dots\} \text{ or } \{a_n\} = \{2n\}.$$

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Sequences can be described by writing rules that specify their terms, such as

$$a_n = \sqrt{n}, \quad b_n = (-1)^{n+1} \frac{1}{n}, \quad c_n = \frac{n-1}{n}, \quad d_n = (-1)^{n+1},$$

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or by listing terms:

$$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\},$$

$$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1} \frac{1}{n}, \dots\right\},$$

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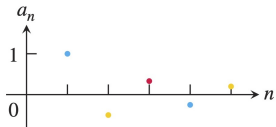
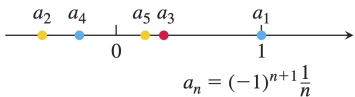
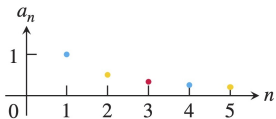
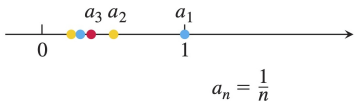
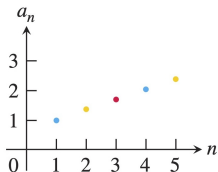
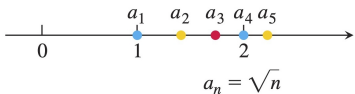
We also sometimes use the notation

$$\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}.$$

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## Representing Sequences

two ways to represent sequences graphically:



# Sequences

## Convergence and Divergence

### Definition.

The sequence  $\{a_n\}$  **converges** to the number  $L$  if for every positive number  $\epsilon$  there corresponds an integer  $N$  such that for all  $n$ ,

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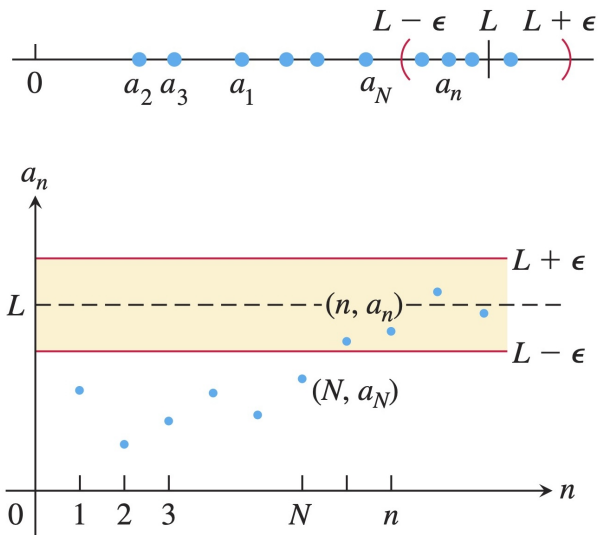
If  $\{a_n\}$  converges to  $L$ , we write  $\lim_{n \rightarrow \infty} a_n = L$  or simply  $a_n \rightarrow L$ , and call  $L$  the **limit** of the sequence.

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**Example.** Show that (a)  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  (b)  $\lim_{n \rightarrow \infty} k = k$  for any constant  $k$

**Solution.**

(a) Let  $\epsilon > 0$  be given.

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Since  $k - k = 0$ , we can use any positive integer  $N$  and the inequality holds for all  $n > N$ . Thus,  $\lim_{n \rightarrow \infty} k = k$ .

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$$\frac{1}{2} < L < \frac{3}{2} \quad \text{and} \quad -\frac{3}{2} < L < -\frac{1}{2}.$$

However, there is no value of  $L$  that can satisfy both of these inequalities simultaneously. Therefore, the sequence cannot converge.

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The sequence  $\{a_n\}$  **diverges to infinity** if for every number  $M$  there exists an integer  $N$  such that for all  $n > N$ ,  $a_n > M$ . If this condition holds, we write

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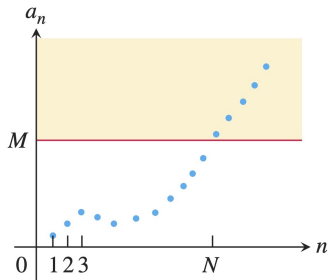
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Similarly if for every number  $m$  there is an integer  $N$  such that for all  $n > N$ ,  $a_n < m$ , we say that  $\{a_n\}$  **diverges to negative infinity** and write

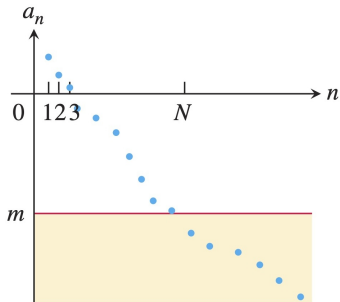
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## Convergence and Divergence



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Let  $\{a_n\}$  and  $\{b_n\}$  be sequences of real numbers, and let  $A$  and  $B$  be real numbers. The following rules hold if  $\lim_{n \rightarrow \infty} a_n = A$  and  $\lim_{n \rightarrow \infty} b_n = B$ :

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4. *Product Rule:*  $\lim_{n \rightarrow \infty} (a_n b_n) = AB$
5. *Quotient Rule:* If  $B \neq 0$ , then  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$

# Sequences

## Calculating Limits of Sequences

**Example.**

$$(a) \lim_{n \rightarrow \infty} \left( -\frac{1}{n} \right)$$

# Sequences

## Calculating Limits of Sequences

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$$(a) \lim_{n \rightarrow \infty} \left( -\frac{1}{n} \right) = -1 \cdot \lim_{n \rightarrow \infty} \frac{1}{n}$$

by constant Multiple Rule

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by difference Rule and from the first example

$$(c) \lim_{n \rightarrow \infty} \left( \frac{5}{n^2} \right) = 5 \lim_{n \rightarrow \infty} \frac{1}{n} \lim_{n \rightarrow \infty} \frac{1}{n}$$

by Product Rule

# Sequences

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$$(d) \lim_{n \rightarrow \infty} \left( \frac{4 - 7n^6}{n^6 + 3} \right) = \lim_{n \rightarrow \infty} \frac{(4/n^6 - 7)}{1 + (3/n^6)} = \frac{0 - 7}{1 + 0} = -7$$

by Sum and Quotient Rules

# Sequences

## Calculating Limits of Sequences

### Theorem. *The Sandwich Theorem for Sequences*

Let  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$  be sequences of real numbers. If  $a_n \leq b_n \leq c_n$  holds for all  $n$  greater than some index  $N$ , and if  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ , then  $\lim_{n \rightarrow \infty} b_n = L$  also.

# Sequences

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# Sequences

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By the Sandwich Theorem, since  $\frac{1}{n} \rightarrow 0$ , we have  $\frac{n!}{n^n} \rightarrow 0$ .

# Sequences

## Calculating Limits of Sequences

*Theorem. The Continuous Function Theorem for Sequences*

Let  $\{a_n\}$  be a sequence of real numbers. If  $a_n \rightarrow L$  and if  $f$  is a function that is continuous at  $L$  and defined at all  $a_n$ , then  $f(a_n) \rightarrow f(L)$ .

# Sequences

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**Example.** Show that  $\sqrt{(n+1)/n} \rightarrow 1$ .

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We know that  $(n+1)/n \rightarrow 1$ . Since  $f(x) = \sqrt{x}$  is continuous, we have  $f((n+1)/n) \rightarrow f(1)$ , or  $\sqrt{(n+1)/n} \rightarrow \sqrt{1} = 1$ .

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**Example.** Show that  $\lim_{n \rightarrow \infty} 2^{1/n} = 1$ .

We know that  $\lim_{n \rightarrow \infty} 1/n = 0$ . Since  $f(x) = 2^x$  is continuous, we have  $\lim_{n \rightarrow \infty} f(1/n) = f(0)$ , or  $\lim_{n \rightarrow \infty} 2^{1/n} = 2^0 = 1$ .

# Sequences

## Calculating Limits of Sequences

### Theorem.

Suppose that  $f(x)$  is a function defined for all  $x \geq n_0$  and that  $\{a_n\}$  is a sequence of real numbers such that  $a_n = f(n)$  for  $n \geq n_0$ . Then

$$\lim_{x \rightarrow \infty} f(x) = L \quad \Rightarrow \quad \lim_{n \rightarrow \infty} a_n = L.$$

# Sequences

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Let  $a_n = \frac{\ln n}{n}$ . If  $f(x) = \frac{\ln x}{x}$ , then  $f(x)$  is a continuous function for  $x \geq 1$  and  $a_n = f(n)$ .

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$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$



L'Hôpital's rule

# Sequences

## Calculating Limits of Sequences

**Example.** Does the sequence whose  $n$ th term is

$$a_n = \left( \frac{n+1}{n-1} \right)^n$$

converge? If so, find  $\lim_{n \rightarrow \infty} a_n$ .

**Solution.** The limit leads to the indeterminate form  $1^\infty$ .

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$$\begin{aligned} \ln a_n &= \ln \left( \frac{n+1}{n-1} \right)^n \\ &= n \ln \left( \frac{n+1}{n-1} \right). \end{aligned}$$

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# Sequences

## Commonly Occuring Limits

Theorem.

The following six sequences converge to the limits listed below:

1. 
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$$5. \quad \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$$

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$$5. \quad \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$$

$$6. \quad \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$$

# Sequences

## Commonly Occuring Limits

### Theorem.

The following six sequences converge to the limits listed below:

$$1. \quad \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

$$2. \quad \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

$$3. \quad \lim_{n \rightarrow \infty} x^{1/n} = 1 \quad (x > 0)$$

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In Formulas (3) through (6),  $x$  remains fixed as  $n \rightarrow \infty$  !

# Sequences

## Commonly Occuring Limits

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$$(f) \quad \frac{100^n}{n!} \longrightarrow 0$$