MAT123 MATHEMATICS I

Lecture 23: Infinite Sequences and Series (Continued)

Outline

Sequences (Continued)

- Bounded Monotonic Sequences

Infinite Series

Sequences (Continued)

Definition.

A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

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A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n. The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

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If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is a **bounded sequence**. If $\{a_n\}$ is not bounded, then it is an **unbounded sequence**.

Example.

(a) The sequence $1, 2, 3, \ldots, n, \ldots$ has no upper bound since it eventually surpasses every number M. However, it is bounded below by every real number less than or equal to 1. The number $m = 1$ is the greatest lower bound for this sequence.

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(b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is bounded above by every number greater than or equal to 1. The number $M = 1$ is the least upper bound for this sequence. It is also bounded below by every number less than or equal to $\frac{1}{2}$. The number $m = \frac{1}{2}$ is the greatest lower bound for this sequence.

Definition.

A sequence $\{a_n\}$ is **nondecreasing** if $a_n \leq a_{n+1}$ for all n. That is, $a_1 \leq a_2 \leq a_3 \leq \cdots$. The sequence is **nonincreasing** if $a_n \geq a_{n+1}$ for all n. The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

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(e) The sequence $1, -1, 1, -1, \ldots$ is not monotonic.

Theorem. *The Monotonic Sequence Theorem*

If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

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- *Monotonicity:*

$$a_{n+1} - a_n = \left(2 - \frac{2}{n+1} - \frac{1}{2^{n+1}}\right) - \left(2 - \frac{2}{n} - \frac{1}{2^n}\right)$$

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If the sequence of partial sums of the series does not converge, then we say that the series **diverges**.

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First:	$s_1 = 1$	1	$2 - 1$
Second:	$s_2 = 1 + \frac{1}{2}$	$\frac{3}{2}$	$2 - \frac{1}{2}$
Third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$\frac{7}{4}$	$2 - \frac{1}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
nth:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$

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"the sum of the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is 2."

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$.

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$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots, \qquad r = 1/2,\ a = 1$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots. \qquad r = -1/3,\ a = 1$$

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If $r = 1$, the nth partial sum of the geometric series is $s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na$, and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$, depending on the sign of a. If $r = -1$, the series diverges because the nth partial sums alternate between a and 0.

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If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1-r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Example. The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9} \left(\frac{1}{3}\right)^{n-1}$$

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$$\frac{a}{1-r} = \frac{5}{1-(-1/4)} = 4.$$

Example. You drop a ball from a meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds to a distance rh, where r is positive but less than 1. Find the total distance the ball travels up and down.

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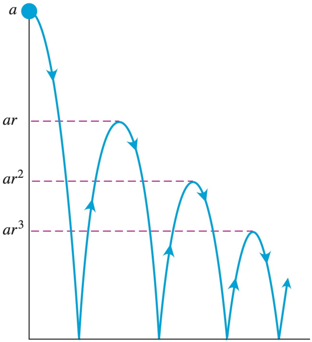

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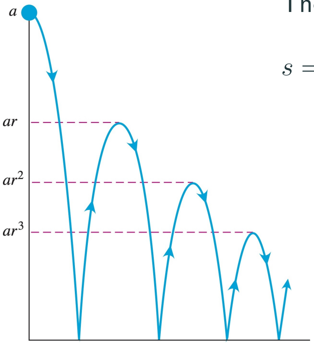

The total distance is

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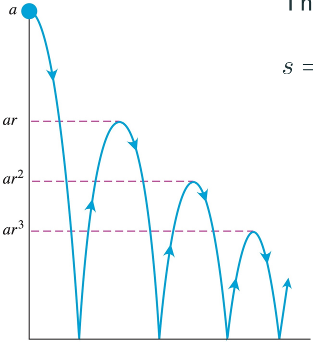

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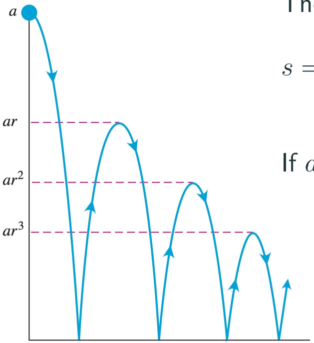

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$$s = a + \underbrace{2ar + 2ar^2 + 2ar^3 + \cdots}_{\text{This sum is } 2ar/(1-r).} = a + \frac{2ar}{1-r} = a\frac{1+r}{1-r}.$$

If $a = 6$ m and $r = 2/3$, for instance, the distance is

$$s = 6\frac{1+(2/3)}{1-(2/3)} = 6\left(\frac{5/3}{1/3}\right) = 30\, m$$

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We now see that $s_k \to 1$ as $k \to \infty$. The series converges, and its sum is 1.

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If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

The nth-Term Test for a Divergent Series

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From this theorem, we get a useful test for divergence:

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \to \infty} a_n$ fails to exist or is different from zero.

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Caution: Remember that $\sum(a_n + b_n)$ can converge when $\sum a_n$ and $\sum b_n$ both diverge. For example, $\sum a_n = 1 + 1 + 1 + \cdots$ and $\sum b_n = (-1) + (-1) + (-1) + \cdots$ both diverge, but $\sum(a_n + b_n) = 0 + 0 + 0 + \cdots$ converges to 0.

Infinite Series

Combining Series

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$$\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}$$

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$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)} \qquad \text{Geometric Series with } a=1 \text{ and } r=1/2, 1/6$$

Example.

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} &= \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} && \text{Difference Rule} \\
&= \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)} && \text{Geometric Series with } a = 1 \text{ and } r = 1/2, 1/6 \\
&= 2 - \frac{6}{5} = \frac{4}{5}.
\end{aligned}
$$

Infinite Series

Adding or Deleting Terms

We can add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence, although in the case of convergence this will usually change the sum. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$ and

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n.$$

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Conversely, if $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \sum_{n=4}^{\infty} \frac{1}{5^n}$$

and

$$\sum_{n=4}^{\infty} \frac{1}{5^n} = \left(\sum_{n=1}^{\infty} \frac{1}{5^n} \right) - \frac{1}{5} - \frac{1}{25} - \frac{1}{125}.$$

Infinite Series

Reindexing

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + a_3 + \cdots$$
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1-h}^{\infty} a_{n+h} = a_1 + a_2 + a_3 + \cdots$$

Reindexing a series means to change the index of summation by adding or subtracting a constant h and adjusting the terms accordingly. This does not change the value of the series, since the sequence of partial sums remains the same.

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Example. We can write the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

as

$$\sum_{n=0}^{\infty} \frac{1}{2^n}, \quad \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}, \quad \text{or even} \quad \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}.$$

The partial sums remain the same no matter what indexing we choose.