MAT123 MATHEMATICS I

Lecture 24: Infinite Sequences and Series (Continued)

Outline

Infinite Series (Continued)

Alternating Series, Absolute and Conditional Convergence

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Infinite Series

Corollary

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

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$$s_{2^k} > \frac{k}{2} \quad \text{for all } k \geq 1 \Longrightarrow \sum \frac{1}{n} \text{ diverges.}$$

The Integral Test

Theorem. *The Integral Test*

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ either both converge or both diverge.

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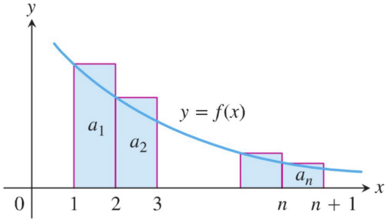
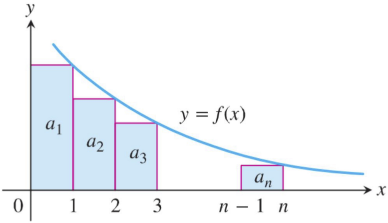


(a)



(b)

The Integral Test

Example. Show that the **p-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges for $p \leq 1$.

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So the series converges by the Integral Test.

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If $p = 1$, we have the (divergent) harmonic series

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In conclusion, we have convergence for $p > 1$ but divergence for all other values of p.

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Solution. Consider the function $f(x) = \dfrac{1}{x^2+1}$.

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Solution. Consider the function $f(x) = \frac{1}{x^2+1}$. This function is continuous, positive, and decreasing for $x \geq 1$. To determine whether the series converges, we evaluate the improper integral

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Since the integral converges, by the Integral Test the series also converges.

Comparison Tests

Theorem. *The Comparison Test*

Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer N

$$d_n \leq a_n \leq c_n \text{ for all } n > N.$$

(a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

Example. Consider the series

$$\sum_{n=1}^{\infty} \frac{5}{5n-1}.$$

Notice that

$$\frac{5}{5n-1} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n}$$

Since the harmonic series $\sum_{n=1}^{\infty}(1/n)$ diverges, by the Comparison Test, part (b), the series also diverges.

Comparison Tests

Example. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

This series converges because its nth term

$$\frac{1}{n^2+1} < \frac{1}{n^2}$$

is less than the nth term of the convergent p-series with $p = 2$.

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Example. Consider the series

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Notice that the terms of this series are all positive and less than or equal to the corresponding terms of

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$$

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which is a convergent geometric series. So by the Comparison Test, the given series converges. We also have

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1-(1/2)} = 3.$$

The fact that 3 is an upper bound for the partial sums of $\sum_{n=0}^{\infty}(1/n!)$ does not mean that the series converges to 3.

Comparison Tests

Example. Test the convergence of the series

$$5 + \frac{2}{3} + \frac{1}{7} + 1 + \frac{1}{2+\sqrt{1}} + \frac{1}{4+\sqrt{2}} + \frac{1}{8+\sqrt{3}} + \cdots + \frac{1}{2^n+\sqrt{n}} + \cdots .$$

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Notice that

$$1 + \frac{1}{2+\sqrt{1}} + \frac{1}{4+\sqrt{2}} + \frac{1}{8+\sqrt{3}} + \cdots$$

$$\leq 1$$

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So by the Comparison Test, the given series converges.

Theorem. *Limit Comparison Test*

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Example. Which of the following series converge, and which diverge?

(a) $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

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Solution. Let $a_n = (2n+1)/(n^2+2n+1)$.

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Solution. Let $a_n=(2n+1)/(n^2+2n+1)$. For large n, we expect a_n to behave like $2n/n^2=2/n$ since the leading terms dominate for large n, so we let $b_n=1/n$.

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and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2,$$

Example. Which of the following series converge, and which diverge?

(a) $\frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\frac{9}{25}+\cdots=\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2}=\sum_{n=1}^{\infty}\frac{2n+1}{n^2+2n+1}$

Solution. Let $a_n=(2n+1)/(n^2+2n+1)$. For large n, we expect a_n to behave like $2n/n^2=2/n$ since the leading terms dominate for large n, so we let $b_n=1/n$. Since

$$\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{2n^2+n}{n^2+2n+1}=2,$$

from the Limit Comparison Test, $\sum a_n$ also diverges.

Example. Which of the following series converge, and which diverge?

(b) $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

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Solution. Let $a_n = 1/(2^n - 1)$.

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Solution. Let $a_n = 1/(2^n - 1)$. For large n, we expect a_n to behave like $1/2^n$, so we let $b_n = 1/2^n$.

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Solution. Let $a_n = 1/(2^n - 1)$. For large n, we expect a_n to behave like $1/2^n$, so we let $b_n = 1/2^n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{and} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - (1/2^n)} = 1,$$

Example. Which of the following series converge, and which diverge?

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$$= \lim_{n\to\infty} \frac{1}{1 - (1/2^n)}$$

$$= 1,$$

from the Limit Comparison Test, $\sum a_n$ also converges.

Example. Which of the following series converge, and which diverge?

(c) $$\frac{1+2\ln 2}{9}+\frac{1+3\ln 3}{14}+\frac{1+4\ln 4}{21}+\cdots=\sum_{n=2}^{\infty}\frac{1+n\ln n}{n^2+5}$$

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Solution. Let $a_n = (1 + n\ln n)/(n^2+5)$.

Example. Which of the following series converge, and which diverge?

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Solution. Let $a_n = (1+n\ln n)/(n^2+5)$. For large n, we expect a_n to behave like $(n\ln n)/n^2 = (\ln n)/n$, which is greater than $1/n$ for $n \geq 3$, so we let $b_n = 1/n$.

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Solution. Let $a_n=(1+n\ln n)/(n^2+5)$. For large n, we expect a_n to behave like $(n\ln n)/n^2=(\ln n)/n$, which is greater than $1/n$ for $n\geq 3$, so we let $b_n=1/n$. Since

$$\sum_{n=2}^{\infty} b_n=\sum_{n=2}^{\infty}\frac{1}{n}\text{ diverges}$$

and

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n+n^2\ln n}{n^2+5}=\infty,$$

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from the Limit Comparison Test, $\sum a_n$ also diverges.

The Ratio and Root Tests

Theorem. *The Ratio Test*

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or $\rho = \infty$,

(c) the test is *inconclusive* if $\rho = 1$.

The Ratio and Root Tests

Example. Investigate the convergence of the following series.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution.

Example. Investigate the convergence of the following series.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution. We apply the Ratio Test to each series.

Example. Investigate the convergence of the following series.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution. We apply the Ratio Test to each series.

(a) Let $a_n = \dfrac{2^n + 5}{3^n}$.

Example. Investigate the convergence of the following series.

$$\text{(a)}\ \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad \text{(b)}\ \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution. We apply the Ratio Test to each series.

(a) Let $a_n = \dfrac{2^n + 5}{3^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} + 5}{3^{n+1}} \cdot \frac{3^n}{2^n + 5}$$

The Ratio and Root Tests

Example. Investigate the convergence of the following series.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution. We apply the Ratio Test to each series.

(a) Let $a_n = \dfrac{2^n + 5}{3^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} + 5}{3^{n+1}} \cdot \frac{3^n}{2^n + 5} = \frac{2^{n+1} + 5}{2^n + 5} \cdot \frac{1}{3}$$

Example. Investigate the convergence of the following series.

$$\text{(a)}\ \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad \text{(b)}\ \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution. We apply the Ratio Test to each series.

(a) Let $a_n = \dfrac{2^n + 5}{3^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} + 5}{3^{n+1}} \cdot \frac{3^n}{2^n + 5} = \frac{2^{n+1} + 5}{2^n + 5} \cdot \frac{1}{3} = \frac{2 + \dfrac{5}{2^n}}{1 + \dfrac{5}{2^n}} \cdot \frac{1}{3} \longrightarrow \frac{2}{1} \cdot \frac{1}{3} = \frac{2}{3}.$$

Example. Investigate the convergence of the following series.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution. We apply the Ratio Test to each series.

(a) Let $a_n = \dfrac{2^n + 5}{3^n}$. Then

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The series converges because $\rho = 2/3$ is less than 1.

Example. Investigate the convergence of the following series.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n+5}{3^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution. We apply the Ratio Test to each series.

(a) Let $a_n = \dfrac{2^n+5}{3^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}+5}{3^{n+1}} \cdot \frac{3^n}{2^n+5} = \frac{2^{n+1}+5}{2^n+5} \cdot \frac{1}{3} = \frac{2+\dfrac{5}{2^n}}{1+\dfrac{5}{2^n}} \cdot \frac{1}{3} \longrightarrow \frac{2}{1} \cdot \frac{1}{3} = \frac{2}{3}.$$

The series converges because $\rho = 2/3$ is less than 1. This does not mean that $2/3$ is the sum of the series. In fact,

$$\sum_{n=0}^{\infty} \frac{2^n+5}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + 5\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{2}{3}} + 5 \cdot \frac{1}{1-\frac{1}{3}} = \frac{21}{2}.$$

Example. Investigate the convergence of the following series.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution. We apply the Ratio Test to each series.

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2(n+1))!}{(n+1)!(n+1)!}$.

Example. Investigate the convergence of the following series.

$$\text{(a)}\ \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad \text{(b)}\ \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution. We apply the Ratio Test to each series.

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2(n+1))!}{(n+1)!(n+1)!}$. Thus,

$$\frac{a_{n+1}}{a_n} = \frac{(2(n+1))!}{(n+1)!(n+1)!} \cdot \frac{n!n!}{(2n)!}$$

Example. Investigate the convergence of the following series.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution. We apply the Ratio Test to each series.

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2(n+1))!}{(n+1)!(n+1)!}$. Thus,

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(2(n+1))!}{(n+1)!(n+1)!} \cdot \frac{n!n!}{(2n)!} \\ &= \frac{(2n+2)(2n+1)(2n)!}{(n+1)(n+1)n!n!} \cdot \frac{n!n!}{(2n)!} \end{aligned}$$

Example. Investigate the convergence of the following series.

$$\text{(a)}\ \sum_{n=0}^{\infty} \frac{2^n+5}{3^n} \qquad \text{(b)}\ \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution. We apply the Ratio Test to each series.

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2(n+1))!}{(n+1)!(n+1)!}$. Thus,

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(2(n+1))!}{(n+1)!(n+1)!} \cdot \frac{n!n!}{(2n)!} \\ &= \frac{(2n+2)(2n+1)(2n)!}{(n+1)(n+1)n!n!} \cdot \frac{n!n!}{(2n)!} \\ &= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n+2}{n+1}\end{aligned}$$

Example. Investigate the convergence of the following series.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution. We apply the Ratio Test to each series.

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2(n+1))!}{(n+1)!(n+1)!}$. Thus,

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(2(n+1))!}{(n+1)!(n+1)!} \cdot \frac{n!n!}{(2n)!} \\
&= \frac{(2n+2)(2n+1)(2n)!}{(n+1)(n+1)n!n!} \cdot \frac{n!n!}{(2n)!} \\
&= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n+2}{n+1} \longrightarrow 4 \text{ as } n \to \infty.
\end{aligned}$$

Example. Investigate the convergence of the following series.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution. We apply the Ratio Test to each series.

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2(n+1))!}{(n+1)!(n+1)!}$. Thus,

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(2(n+1))!}{(n+1)!(n+1)!} \cdot \frac{n!n!}{(2n)!} \\
&= \frac{(2n+2)(2n+1)(2n)!}{(n+1)(n+1)n!n!} \cdot \frac{n!n!}{(2n)!} \\
&= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n+2}{n+1} \longrightarrow 4 \text{ as } n \to \infty.
\end{aligned}$$

Since $\rho = 4 > 1$, the series diverges.

Theorem. *The Root Test*

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

Then **(a)** the series *converges* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or $\rho = \infty$, and **(c)** the test is *inconclusive* if $\rho = 1$.

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(a) Let $a_n = \dfrac{n^2}{2^n}$.

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(a) Let $a_n = \dfrac{n^2}{2^n}$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{n^2}{2^n}}$$

The Ratio and Root Tests

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(a) Let $a_n = \dfrac{n^2}{2^n}$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{2}$$

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(a) Let $a_n = \dfrac{n^2}{2^n}$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{2} = \frac{(\sqrt[n]{n})^2}{2}$$

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(a) Let $a_n = \dfrac{n^2}{2^n}$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{2} = \frac{(\sqrt[n]{n})^2}{2} \longrightarrow \frac{1}{2}.$$

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(a) Let $a_n = \dfrac{n^2}{2^n}$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{2} = \frac{(\sqrt[n]{n})^2}{2} \longrightarrow \frac{1}{2}.$$

Since $\rho = \frac{1}{2} < 1$, the series converges.

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(b) Let $a_n = \dfrac{2^n}{n^3}$.

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(b) Let $a_n = \dfrac{2^n}{n^3}$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{2^n}{n^3}}$$

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(b) Let $a_n = \dfrac{2^n}{n^3}$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{\sqrt[n]{n^3}}$$

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(b) Let $a_n = \dfrac{2^n}{n^3}$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{\sqrt[n]{n^3}} = 2 \cdot \frac{1}{(\sqrt[n]{n})^3}$$

The Ratio and Root Tests

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(b) Let $a_n = \dfrac{2^n}{n^3}$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{\sqrt[n]{n^3}} = 2 \cdot \frac{1}{(\sqrt[n]{n})^3} \longrightarrow 2.$$

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(b) Let $a_n = \dfrac{2^n}{n^3}$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{\sqrt[n]{n^3}} = 2 \cdot \frac{1}{(\sqrt[n]{n})^3} \longrightarrow 2.$$

Since $\rho = 2 > 1$, the series diverges.

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(c) Let $a_n = \left(\dfrac{1}{1+n}\right)^n$.

Example. Which of the following series converge, and which diverge?

$$\text{(a)}\ \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)}\ \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)}\ \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(c) Let $a_n = \left(\frac{1}{1+n}\right)^n$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{1}{1+n}\right)^n}$$

Example. Which of the following series converge, and which diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution. We apply the Root Test to each series.

(c) Let $a_n = \left(\frac{1}{1+n}\right)^n$. Then

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The Ratio and Root Tests

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Since $\rho = 0 < 1$, the series converges.

Alternating Series, Absolute and Conditional Convergence

Alternating Series, Absolute and Conditional Convergence

A series in which the terms are alternately positive and negative is an **alternating series**. Here are three examples:

- $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots.$
- $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{2^n-1} = -\frac{1}{1} + \frac{1}{3} - \frac{1}{7} + \frac{1}{15} - \frac{1}{31} + \cdots.$
- $\sum_{n=1}^{\infty} n\cos(\pi(n-1)) = \sum_{n=1}^{\infty} n\,(-1)^{n-1} = 1 - 2 + 3 - 4 + 5 - \cdots.$

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We see from these examples that the terms of an alternating series can be written in the form

$$a_n = (-1)^{n+1}u_n \quad \text{or} \quad a_n = (-1)^n u_n$$

where $u_n = |a_n| > 0$ for all n.

Alternating Series, Absolute and Conditional Convergence

Theorem. *The Alternating Series Test (Leibniz's Test)*

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $\lim_{n\to\infty} u_n = 0$.

Example. Determine whether the alternating harmonic series

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Solution. Let $u_n = \frac{1}{n}$. Then the u_n's are all positive. Also, the sequence $\{u_n\}$ is nonincreasing because

$$u_n = \frac{1}{n} \geq \frac{1}{n+1} = u_{n+1} \text{ for all } n \geq 1.$$

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Finally, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$. Since all three conditions of the Alternating Series Test are satisfied, the series converges.

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$$f'(x) = \frac{1 - \ln x}{x^2} < 0 \text{ for } x \geq e.$$

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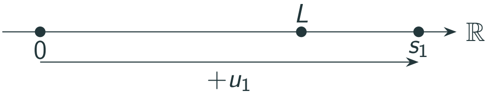
Thus, $f(x)$ is decreasing for $x \geq e$, which means that the sequence $\{u_n\}$ is decreasing for $n \geq 3$. Finally, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\ln n}{n} = 0$ by L'Hôpital's Rule. In conclusion, by the Alternating Series Test, the series converges.

Consider a convergent alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n, \qquad s_n = \sum_{k=1}^{n}(-1)^{k+1}u_k, \qquad \lim_{n\to\infty} s_n = L.$$



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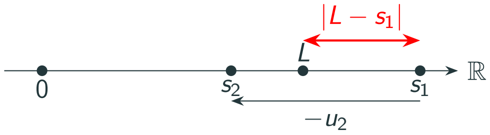

$$|L - s_1| \; < \; |u_2|.$$

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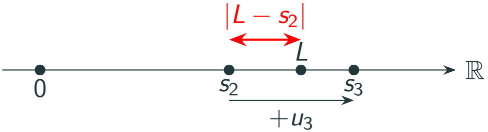

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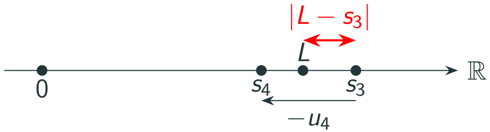

$$|L - s_3| \ < \ |u_4|.$$

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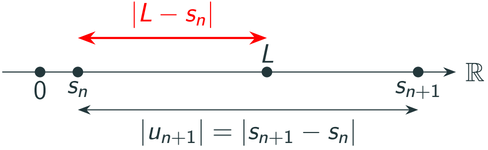

$$|L - s_n| \; < \; |u_{n+1}| \qquad \text{(alternating series with } u_n \downarrow 0\text{)}.$$

Theorem. *Alternating Series Estimation Theorem*

Suppose that the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the conditions of the Alternating Series Test. Then for $n \geq N$,

$$s_n = u_1 - u_2 + u_3 - \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is at most u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder $R_n = L - s_n$ has the same sign as the first unused term $(-1)^{n+2}u_{n+1}$.

Example. Let us try the preceding theorem on a series whose sum we know.

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} - \cdots .$$

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Since this is a geometric series with ratio $r = -\frac{1}{2}$, its sum is

$$L = \frac{1}{1-(-\frac{1}{2})} = 2/3.$$

Example. Let us try the preceding theorem on a series whose sum we know.

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\frac{1}{256}-\cdots.$$

Since this is a geometric series with ratio $r=-\frac{1}{2}$, its sum is

$$L=\frac{1}{1-(-\frac{1}{2})}=2/3.$$

The eighth partial sum is

$$S_8=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}=\frac{170}{256}=0.6640625.$$

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According to the Alternating Series Estimation Theorem, the error in using S_8 to approximate L is at most the absolute value of the first unused term, $1/256$. Indeed,

$$|R_8| = |L - S_8| = \left|\frac{2}{3} - 0.6640625\right| = 0.0026041666\ldots < \frac{1}{256} = 0.00390625.$$

Definition. *Absolute and Conditional Convergence*

A series $\sum a_n$ is said to be **absolutely convergent** if the series of absolute values $\sum |a_n|$ converges. If $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ is said to be **conditionally convergent**.

Example. Determine whether the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}$$

is absolutely convergent, conditionally convergent, or divergent.

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Solution. Consider the series of absolute values

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$$\sum_{n=0}^{\infty}\left|(-1)^n\frac{1}{2^n}\right| = \sum_{n=0}^{\infty}\frac{1}{2^n}.$$

This series converges (it is a geometric series with ratio $r = \frac{1}{2} < 1$).

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$$\sum_{n=0}^{\infty}\left|(-1)^n \frac{1}{2^n}\right| = \sum_{n=0}^{\infty}\frac{1}{2^n}.$$

This series converges (it is a geometric series with ratio $r = \frac{1}{2} < 1$). Since the series of absolute values converges, the original series is absolutely convergent.

Example. Determine whether the alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

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This series diverges (it is the harmonic series).

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This series diverges (it is the harmonic series). Since the original series converges (by the Alternating Series Test) but the series of absolute values diverges, the alternating harmonic series is conditionally convergent.

Theorem. *Absolute Convergence Test*

If a series $\sum a_n$ is absolutely convergent, then it is convergent. That is, if $\sum |a_n|$ converges, then $\sum a_n$ also converges.

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

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Solution. Notice that this series contains both positive and negative terms, but it is not an alternating series because the signs of the terms do not alternate.

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$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}.$$

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$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}.$$

Since $|\sin n| \leq 1$ for all n, we have

$$0 \leq \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}.$$

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The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$). By the Comparison Test, the series of absolute values converges.

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is absolutely convergent, conditionally convergent, or divergent.

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The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$). By the Comparison Test, the series of absolute values converges. Thus the original series also converges.