MAT123 MATHEMATICS I

Lecture 25: Infinite Sequences and Series (Continued)

Outline

- Power Series
  - The Radius of Convergence of a Power Series
  - Differentiation and Integration of Power Series
  - Operations on Power Series

- Taylor and Maclaurin Series
  - Taylor Series for Some Elementary Functions
  - Other Maclaurin and Taylor Series

Power Series

Power Series

Definition

A **power series about** $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

A **power series about** $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

in which the **center** a and the **coefficients** c_n are constants.

Power Series

Example. Consider the following power series about $x = 0$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots .$$

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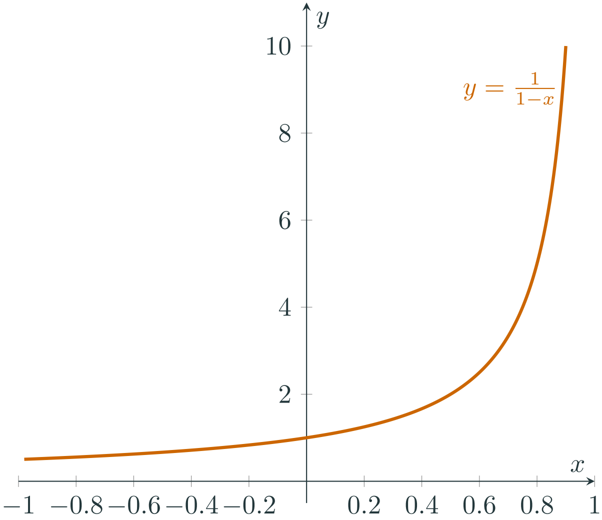
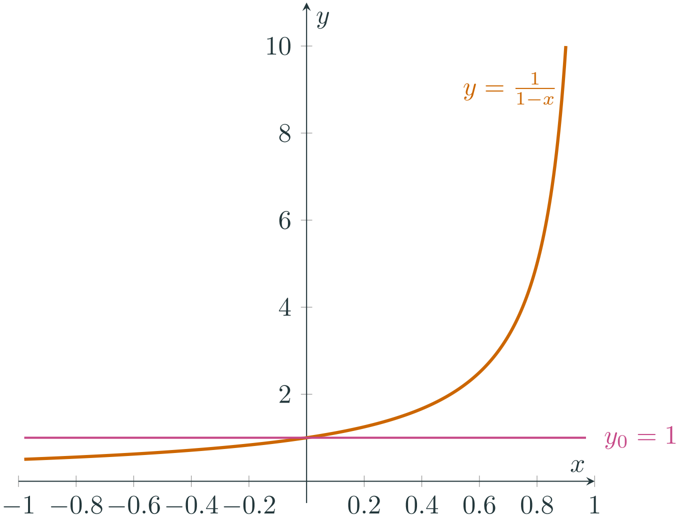
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

Power Series

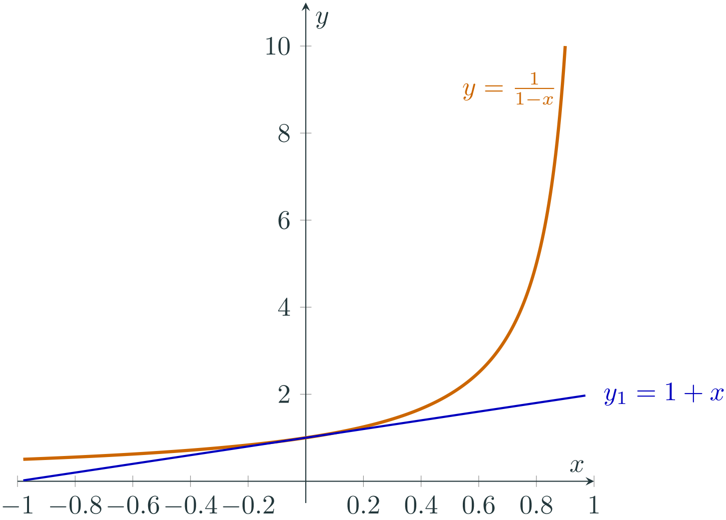


Power Series



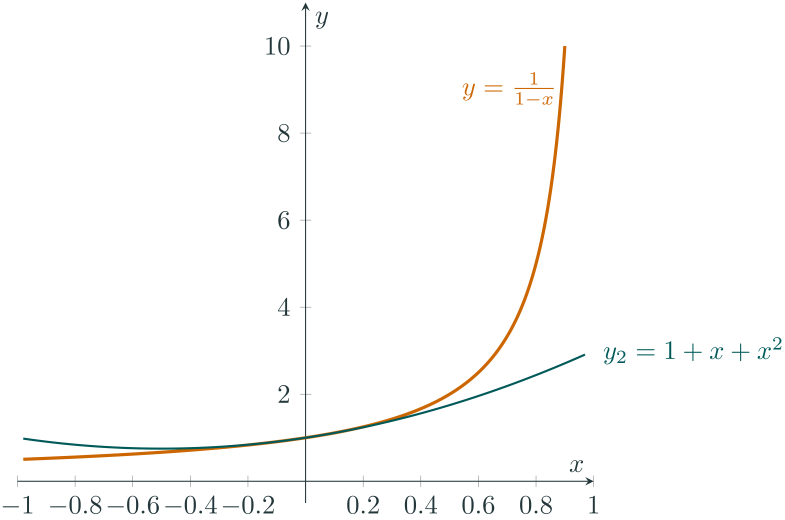
$$\frac{1}{1-x} \approx 1 \qquad (|x| < 1)$$

Power Series



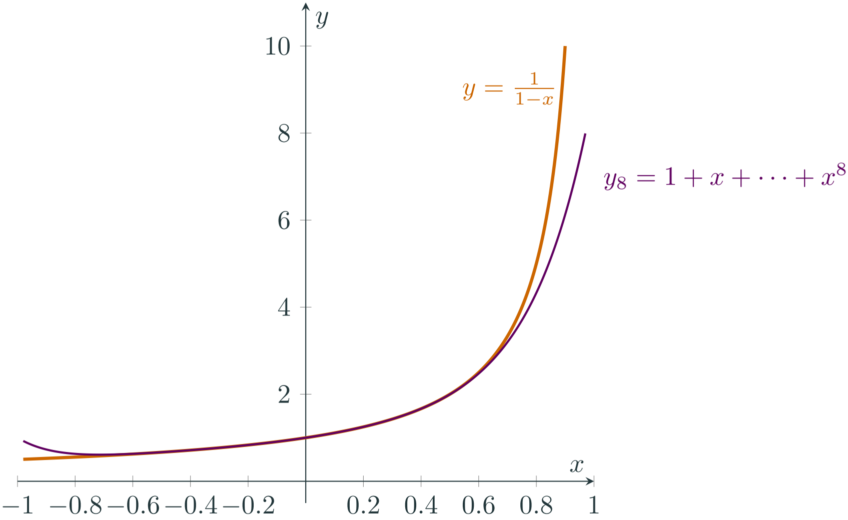
$$\frac{1}{1-x} \approx 1 + x \qquad (|x| < 1)$$

Power Series



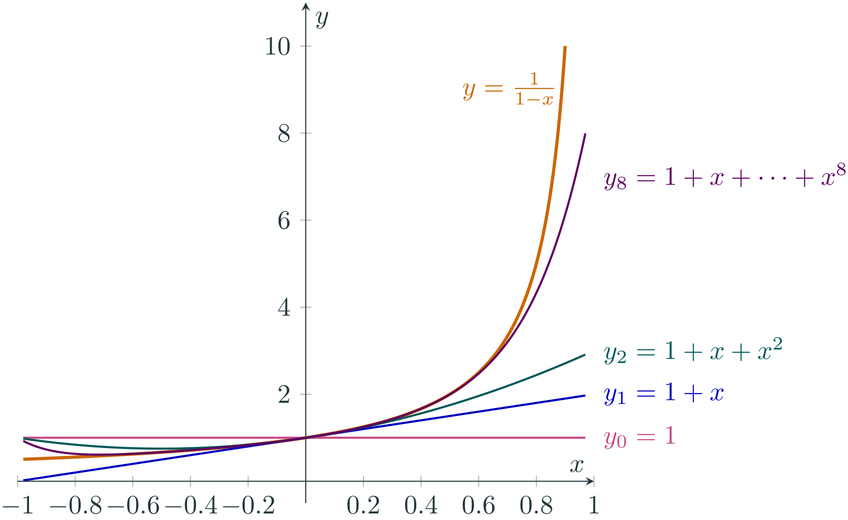
$$\frac{1}{1-x} \approx 1 + x + x^2 \qquad (|x| < 1)$$

Power Series



$$\frac{1}{1-x} \approx 1 + x + x^2 + \cdots + x^8 \qquad (|x| < 1)$$

Power Series



$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^8 + \cdots + x^n + \cdots \qquad (|x| < 1)$$

Power Series

Example. Consider the following power series:

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots .$$

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Notice that $a = 2$, and the coefficients are given by $c_n = \left(-\frac{1}{2}\right)^n$.

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$$\left|-\frac{1}{2}(x-2)\right| < 1 \quad \Rightarrow \quad |x-2| < 2 \quad \Rightarrow \quad 0 < x < 4,$$

and diverges if $|x-2| \geq 2$.

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$$1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-2)^n+\cdots.$$

Notice that $a=2$, and the coefficients are given by $c_n=\left(-\frac{1}{2}\right)^n$. This is a geometric series with first term 1 and ratio $r=-\frac{1}{2}(x-2)$. The series converges if

$$\left|-\frac{1}{2}(x-2)\right|<1 \quad\Rightarrow\quad |x-2|<2 \quad\Rightarrow\quad 0<x<4,$$

and diverges if $|x-2|\geq 2$. When it converges, its sum is given by

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n(x-2)^n=\frac{1}{1+\frac{1}{2}(x-2)}=\frac{2}{x},\quad 0<x<4.$$

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$$\textbf{(a)}\ \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

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Solution. Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the power series in question.

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So the given power series converges for $-1 < x \leq 1$ and diverges elsewhere.

Power Series

Example. For what values of x do the following power series converge?

$$\textbf{(b)}\quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Solution. Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the power series in question.

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By the Ratio Test, the series converges absolutely if $L = x^2 < 1$ and diverges if $L = x^2 > 1$. Now we need to check the endpoints $x = -1$ and $x = 1$.

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So the given power series converges for $-1 \le x \le 1$ and diverges elsewhere.

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Example. For what values of x do the following power series converge?

$$\textbf{(c)}\ \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Solution. Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the power series in question.

(c) Let $u_n = \frac{x^n}{n!}$.

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Solution. Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the power series in question.

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$$L = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

Power Series

Example. For what values of x do the following power series converge?

$$\textbf{(c)}\ \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

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By the Ratio Test, the series converges absolutely for all x.

Power Series

Example. For what values of x do the following power series converge?

$$\textbf{(d)}\ \sum_{n=0}^{\infty} n!\,x^n = 1 + x + 2!\,x^2 + 3!\,x^3 + \cdots$$

Solution. Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the power series in question.

(d) Let $u_n = n!\,x^n$.

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Example. For what values of x do the following power series converge?

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$$L = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\,x^{n+1}}{n!\,x^n} \right|$$

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By the Ratio Test, the series diverges for all $x \neq 0$. For $x = 0$, the series is just 1.

Power Series

Theorem. *The Convergence Theorem for Power Series*

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges at a point $x = c \neq 0$, then it converges absolutely for every x such that $|x| < |c|$. If the series diverges at a point $x = d$, then it diverges for every x such that $|x| > |d|$.

Power Series

The Radius of Convergence of a Power Series

Corollary

The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

- There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$ The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
- The series converges absolutely for all x $(R = \infty)$.
- The series converges at $x = a$ and diverges elsewhere $(R = 0)$.

Power Series

The Radius of Convergence of a Power Series

Corollary

The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

- There is a positive number R such that the series diverges for x with $|x-a|>R$ but converges absolutely for x with $|x-a|<R$ The series may or may not converge at either of the endpoints $x=a-R$ and $x=a+R$.
- The series converges absolutely for all x $(R=\infty)$.
- The series converges at $x=a$ and diverges elsewhere $(R=0)$.

The number R is called the **radius of convergence** of the power series, and the interval of radius R centered at $x=a$ is called the **interval of convergence**. If the series converges for all values of x, we say its radius of convergence is infinite. If it converges only at $x=a$, we say its radius of convergence is zero.

Power Series

The Radius of Convergence of a Power Series

Example. Determine the centre, radius, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{n^2+1} \left(x+\frac{5}{2}\right)^n .$$

Solution.

Power Series

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Solution. Let $a_n = \dfrac{(2x+5)^n}{(n^2+1)3^n}$.

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Example. Determine the centre, radius, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{n^2+1} \left(x+\frac{5}{2}\right)^n .$$

Solution. Let $a_n = \dfrac{(2x+5)^n}{(n^2+1)3^n}$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n^2+1)3^n}{(n^2+2n+2)3^{n+1}}|2x+5|$$

Power Series

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Solution. Let $a_n = \dfrac{(2x+5)^n}{(n^2+1)3^n}$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n^2+1)3^n}{(n^2+2n+2)3^{n+1}}|2x+5| \longrightarrow \frac{1}{3}|2x+5|.$$

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$$\frac{1}{3}|2x+5| < 1$$

Power Series

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Solution. Let $a_n = \dfrac{(2x+5)^n}{(n^2+1)3^n}$. Then,

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Power Series

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Example. Determine the centre, radius, and interval of convergence of

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Solution. Let $a_n = \dfrac{(2x+5)^n}{(n^2+1)3^n}$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n^2+1)3^n}{(n^2+2n+2)3^{n+1}} |2x+5| \longrightarrow \frac{1}{3}|2x+5|.$$

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$$\frac{1}{3}|2x+5| < 1 \quad \Rightarrow \quad |2x+5| < 3 \quad \Rightarrow \quad -4 < x < -1,$$

Power Series

The Radius of Convergence of a Power Series

Example. Determine the centre, radius, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{n^2+1} \left(x + \frac{5}{2}\right)^n .$$

Solution. Let $a_n = \dfrac{(2x+5)^n}{(n^2+1)3^n}$. Then,

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By the Ratio Test, the series converges if

$$\frac{1}{3}|2x+5| < 1 \quad \Rightarrow \quad |2x+5| < 3 \quad \Rightarrow \quad -4 < x < -1,$$

and it diverges if $x > -1$ or $x < -4$.

Power Series

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Example. Determine the centre, radius, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{n^2+1} \left(x+\frac{5}{2}\right)^n.$$

Solution.

We need to check separately the endpoints $x = -4$ and $x = -1$:

Power Series

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Solution.

We need to check separately the endpoints $x = -4$ and $x = -1$:

- At $x = -4$, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$, which converges by the Alternating Series Test.

Example. Determine the centre, radius, and interval of convergence of

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- At $x = -1$, the series becomes $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$, which converges by Comparison test with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $(p = 2 > 1)$.

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$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{n^2+1} \left(x+\frac{5}{2}\right)^n .$$

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Thus, the series converges for $-4 \leq x \leq -1$ and diverges elsewhere.

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Solution.

We need to check separately the endpoints $x=-4$ and $x=-1$:

- At $x=-4$, the series becomes $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$, which converges by the Alternating Series Test.
- At $x=-1$, the series becomes $\sum\limits_{n=0}^{\infty} \frac{1}{n^2+1}$, which converges by Comparison test with the p-series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ $(p=2>1)$.

Thus, the series converges for $-4 \le x \le -1$ and diverges elsewhere. Therefore, the centre is $-5/2$, the radius is $3/2$, and the interval of convergence is $[-4,-1]$.

Power Series

Differentiation and Integration of Power Series

Theorem. *Term-by-Term Differentiation Theorem*

If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad \text{on the interval} \quad a - R < x < a + R.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}, \quad f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}, \quad \ldots$$

Each of these derived series converges at every point of the interval $a - R < x < a + R$.

Power Series

Differentiation and Integration of Power Series

Theorem. *Term-by-Term Integration Theorem*

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a - R < x < a + R$ $(R > 0)$. Then, for every x in this interval,

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges, and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

Differentiation and Integration of Power Series

Example. Find power series representations for the functions

$$\textbf{(a)}\ \frac{1}{(1-x)^2} \quad \textbf{(b)}\ \frac{1}{(1-x)^3} \quad \textbf{(c)}\ \ln(1+x)$$

by starting with the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad (-1 < x < 1)$$

and using differentiation, integration, and substitution as necessary. Where is each series valid?

Power Series

Differentiation and Integration of Power Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad (-1 < x < 1)$$

Solution.

Power Series

Differentiation and Integration of Power Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad (-1 < x < 1)$$

Solution.

(a)

Power Series

Differentiation and Integration of Power Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad (-1 < x < 1)$$

Solution.

(a) Differentiate the above geometric series term by term to obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad (-1 < x < 1).$$

(b)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad (-1 < x < 1)$$

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(a) Differentiate the above geometric series term by term to obtain

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(b) Differentiate the result in (a) term by term to obtain

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-2} = 2 + 6x + 12x^2 + 20x^3 + \cdots \quad (-1 < x < 1).$$

Power Series

Differentiation and Integration of Power Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad (-1 < x < 1)$$

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$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-2} = 2 + 6x + 12x^2 + 20x^3 + \cdots \quad (-1 < x < 1).$$

Now divide by 2:

$$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2} = 1 + 3x + 6x^2 + 10x^3 + \cdots \quad (-1 < x < 1).$$

Power Series

Differentiation and Integration of Power Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad (-1 < x < 1)$$

Solution.

(c)

Power Series

Differentiation and Integration of Power Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad (-1 < x < 1)$$

Solution.

(c) Substitute $-t$ in place of x in the original geometric series:

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-t)^n = 1 - t + t^2 - t^3 + \cdots \quad (-1 < t < 1)$$

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$$\ln(1+x) = \int_0^x \frac{1}{1+t}\, dt$$

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Power Series

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$$\frac{1}{1+t}=\sum_{n=0}^{\infty}(-t)^n = 1-t+t^2-t^3+\cdots \quad (-1<t<1)$$

Integrating from 0 to x, where $|x|<1$, gives

$$\begin{aligned}\ln(1+x) &= \int_0^x \frac{1}{1+t}\,dt = t-t^2+\frac{t^3}{3}-\cdots\,\Big|_0^x \\ &= x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\end{aligned}$$

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Integrating from 0 to x, where $|x| < 1$, gives

$$\begin{aligned} \ln(1+x) &= \int_0^x \frac{1}{1+t}\,dt = t - t^2 + \frac{t^3}{3} - \cdots \Bigg|_0^x \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \end{aligned}$$

or, equivalently,

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x < 1.$$

Power Series

Differentiation and Integration of Power Series

Example. Use the geometric series of the previous example to find a power series representation for $\tan^{-1}(x)$.

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Example. Use the geometric series of the previous example to find a power series representation for $\tan^{-1}(x)$.

Solution. Substitute $-t^2$ for x in the geometric series. Since $0 \leq t^2 < 1$ for $|t| < 1$, we have

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n} = 1 - t^2 + t^4 - t^6 + \cdots \quad (-1 < t < 1).$$

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Now integrate from 0 to x, where $|x| < 1$:

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2}\, dt = \int_0^x (1 - t^2 + t^4 - t^6 + t^8 - \cdots) dt$$

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Now integrate from 0 to x, where $|x| < 1$:

$$\begin{aligned} \tan^{-1}(x) &= \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x (1 - t^2 + t^4 - t^6 + t^8 - \cdots)dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \end{aligned}$$

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Example. Use the geometric series of the previous example to find a power series representation for $\tan^{-1}(x)$.

Solution. Substitute $-t^2$ for x in the geometric series. Since $0 \le t^2 < 1$ for $|t| < 1$, we have

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n} = 1 - t^2 + t^4 - t^6 + \cdots \quad (-1 < t < 1).$$

Now integrate from 0 to x, where $|x| < 1$:

$$\begin{aligned}\tan^{-1}(x) &= \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x (1 - t^2 + t^4 - t^6 + t^8 - \cdots)dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad (-1 < x < 1).\end{aligned}$$

Power Series

Operations on Power Series

Example. Find a series representation of $f(x) = 1/(2+x)$ in powers of $x - 1$. What is the interval of convergence of this series?

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$$\frac{1}{2+x} = \frac{1}{3+t}$$

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\frac{1}{2+x} &= \frac{1}{3+t} = \frac{1}{3}\cdot\frac{1}{1+\frac{t}{3}} \\
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&= \sum_{n=0}^{\infty}(-1)^n\frac{t^n}{3^{n+1}} \quad (-3 < t < 3) \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^n}{3^{n+1}} \quad (-2 < x < 4).
\end{aligned}$$

Taylor and Maclaurin Series

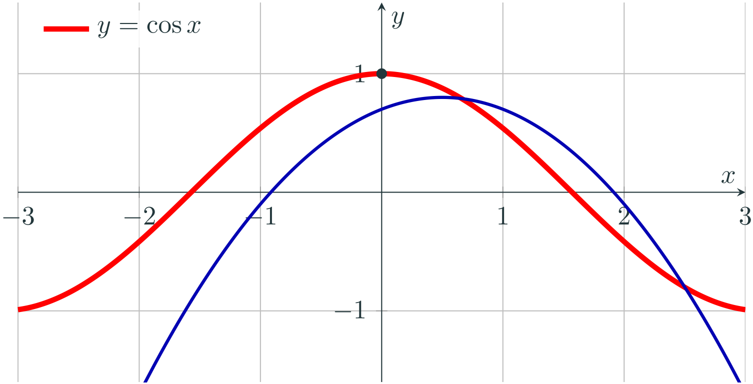
Taylor and Maclaurin Series

The main idea

To approximate a (possibly nonpolynomial) function f near a point a by a polynomial.

Why? Because polynomials are easy to compute, manipulate, differentiate, and integrate.

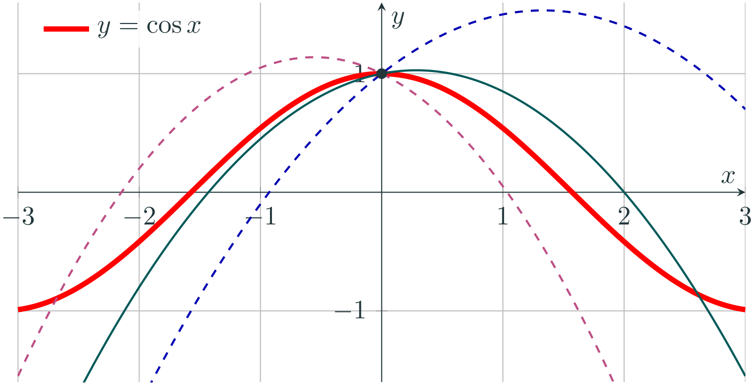
An example: Quadratic approximation of $\cos x$ near 0



What is the best quadratic approximation to $\cos x$ near 0?

$P(x) = a_0 + a_1 x + a_2 x^2$ (we want P to match $\cos x$ near 0)

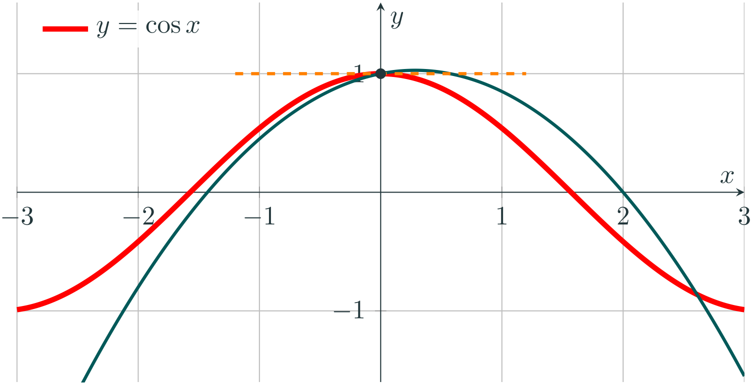
An example: Quadratic approximation of $\cos x$ near 0



First constraint: $P(0) = \cos 0 = 1 \Rightarrow a_0 = 1.$

Now many quadratics pass through $(0, 1)$. **Which is best near** 0?

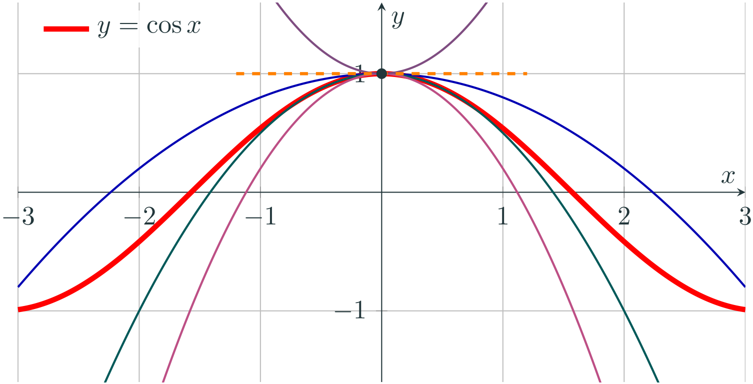
An example: Quadratic approximation of $\cos x$ near 0



To be better near 0, it should share the same tangent (same slope) at 0.

$$\cos'(0) = 0 \quad \Rightarrow \quad \text{tangent line is } y = 1.$$

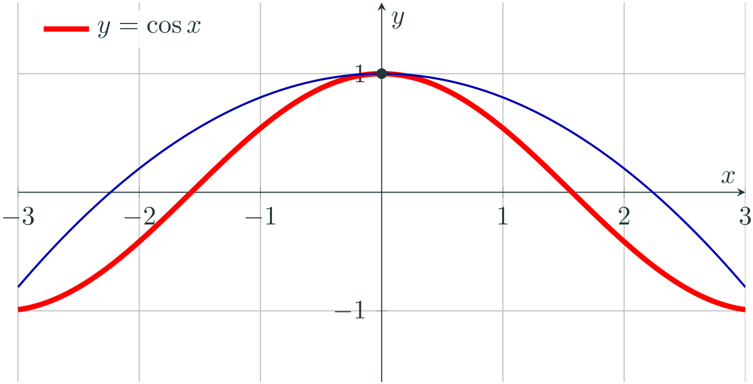
An example: Quadratic approximation of $\cos x$ near 0



Second constraint: $P'(0) = \cos'(0) = 0 \;\Rightarrow\; a_1 = 0.$

Still many choices (different curvature). **Which is best now?**

An example: Quadratic approximation of $\cos x$ near 0

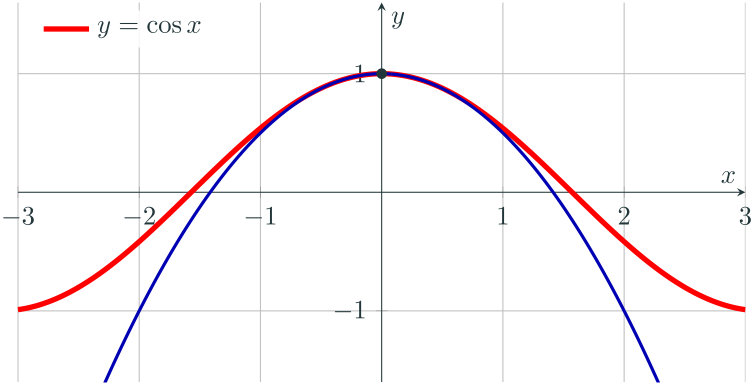


To have an even better approximation near 0,

it should curve at the same rate as $\cos x$ near 0.

$$\cos''(0) = -\cos(0) = -1$$

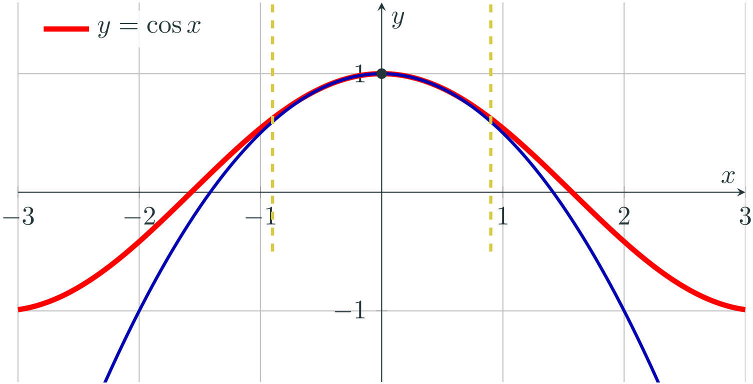
An example: Quadratic approximation of $\cos x$ near 0



Third constraint: $P''(0) = \cos''(0) = -1.$

$$P''(x) = 2a_2 \Rightarrow 2a_2 = -1 \Rightarrow a_2 = -\frac{1}{2} \quad \Longrightarrow \quad \boxed{P_2(x) = 1 - \frac{x^2}{2}}.$$

An example: Quadratic approximation of $\cos x$ near 0



$P_2(x) = 1 - \dfrac{x^2}{2}$ is the best quadratic approximation to $\cos x$ near 0.

Note that the coefficients a_0, a_1, and a_2 are determined by 0th, 1st, and 2nd derivatives of $\cos x$ at 0, respectively.

Now we generalize this idea!

Taylor and Maclaurin Series

Theorem

Suppose the series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

converges to a function $f(x)$ for $c - R < x < c + R$, where $R > 0$. Then

$$a_k = \frac{f^{(k)}(c)}{k!} \quad \text{for } k = 0, 1, 2, \ldots .$$

Taylor and Maclaurin Series

Here is why the coefficients are given by the formula in the theorem:

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots \\
f'(x) &= \sum_{n=1}^{\infty} na_n(x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots \\
f''(x) &= \sum_{n=2}^{\infty} n(n-1)a_n(x-c)^{n-2} = 2a_2 + 6a_3(x-c) + \cdots \\
&\vdots \\
f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)a_n(x-c)^{n-k} \\
&= k!a_k + \frac{(k+1)!}{1!}a_{k+1}(x-c) + \frac{(k+2)!}{2!}a_{k+2}(x-c)^2 + \cdots\ .
\end{aligned}
$$

Each series converges for $c - R < x < c + R$. Setting $x = c$ gives $f^{(k)}(c) = k!a_k$, or equivalently, $a_k = f^{(k)}(c)/k!$.

Taylor and Maclaurin Series

Definition. *Taylor and Maclaurin Series*

If $f(x)$ has derivatives of all orders at $x = c$, then the power series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

is called the **Taylor series of f about c** (or the **Taylor series of f in powers of $x-c$**). In the special case of $c = 0$, the series is usually called the **Maclaurin series of f**.

Taylor and Maclaurin Series

Definition. *Analytic Functions*

A function f is said to be **analytic at a point** c if f has a Taylor series at c and that series converges to $f(x)$ in an open interval containing c. If f is analytic at each point of an open interval, then we say that f is analytic on that interval.

Taylor and Maclaurin Series

Taylor Series for Some Elementary Functions

Example. Find the Taylor series for $f(x) = e^x$ about $x = c$. Where does the series converge to e^x? Where is e^x analytic? What is the Maclaurin series for e^x?

Solution.

Taylor and Maclaurin Series

Taylor Series for Some Elementary Functions

Example. Find the Taylor series for $f(x) = e^x$ about $x = c$. Where does the series converge to e^x? Where is e^x analytic? What is the Maclaurin series for e^x?

Solution. Since all derivatives of $f(x) = e^x$ are equal to e^x, we have

$$f^{(k)}(c) = e^c \quad \text{for } k = 0, 1, 2, \ldots.$$

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$$f^{(k)}(c) = e^c \quad \text{for } k = 0, 1, 2, \ldots.$$

Thus, the Taylor series for e^x about $x = c$ is

$$\sum_{k=0}^{\infty} \frac{e^c}{k!}(x-c)^k = e^c \left[1 + \frac{(x-c)}{1!} + \frac{(x-c)^2}{2!} + \frac{(x-c)^3}{3!} + \cdots \right].$$

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To determine where the series converges to e^x, we use the Ratio Test:

Taylor and Maclaurin Series

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$$L = \lim_{k\to\infty} \left| \frac{e^c/(k+1)!(x-c)^{k+1}}{e^c/k!(x-c)^k} \right|$$

Taylor and Maclaurin Series

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Taylor and Maclaurin Series

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Thus, the series converges for all x, i.e. $R = \infty$.

Taylor and Maclaurin Series

Taylor Series for Some Elementary Functions

Example. Find the Taylor series for $f(x) = e^x$ about $x = c$. Where does the series converge to e^x? Where is e^x analytic? What is the Maclaurin series for e^x?

Solution. Suppose the sum is $g(x)$:

$$g(x) = \sum_{k=0}^{\infty} \frac{e^c}{k!}(x-c)^k = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \cdots .$$

Taylor and Maclaurin Series

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Differentiating term by term gives

$$g'(x) = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \cdots = g(x).$$

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Differentiating term by term gives

$$g'(x) = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \cdots = g(x).$$

Since $g'(x) = g(x)$, we have $g(x) = ke^x$ for some real number k. We know that $g(c) = e^c$, so $k = 1$. Thus, we conclude that $g(x) = e^x$.

Taylor and Maclaurin Series

Taylor Series for Some Elementary Functions

Example. Find the Taylor series for $f(x) = e^x$ about $x = c$. Where does the series converge to e^x? Where is e^x analytic? What is the Maclaurin series for e^x?

Solution. The Taylor series for e^x about $x = c$ converges to e^x for every real number x:

$$e^x = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n.$$

Taylor and Maclaurin Series

Taylor Series for Some Elementary Functions

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$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{(for all } x\text{)}$$

Taylor and Maclaurin Series

Taylor Series for Some Elementary Functions

Example. Find the Maclaurin series for (a) $\sin x$, (b) $\cos x$. Where does each series converge?

Taylor and Maclaurin Series

Taylor Series for Some Elementary Functions

Example. Find the Maclaurin series for (a) $\sin x$, (b) $\cos x$. Where does each series converge?

Solution. (a) Let $f(x) = \sin x$.

Taylor and Maclaurin Series

Taylor Series for Some Elementary Functions

Example. Find the Maclaurin series for (a) $\sin x$, (b) $\cos x$. Where does each series converge?

Solution. (a) Let $f(x) = \sin x$. Then we have $f(0) = 0$ and

$$\begin{aligned} f'(x) &= \cos x, & f'(0) &= 1, \\ f''(x) &= -\sin x, & f''(0) &= 0, \\ f^{(3)}(x) &= -\cos x, & f^{(3)}(0) &= -1, \\ f^{(4)}(x) &= \sin x, & f^{(4)}(0) &= 0, \\ f^{(5)}(x) &= \cos x, & f^{(5)}(0) &= 1. \\ &\vdots & &\vdots \end{aligned}$$

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Thus the Maclaurin series for $\sin x$ is

$$g(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Taylor and Maclaurin Series

Taylor Series for Some Elementary Functions

Example. Find the Maclaurin series for (a) $\sin x$, (b) $\cos x$. Where does each series converge?

Solution. (a) To determine where the series converges, we use the Ratio Test:

Taylor and Maclaurin Series

Taylor Series for Some Elementary Functions

Example. Find the Maclaurin series for (a) $\sin x$, (b) $\cos x$. Where does each series converge?

Solution. (a) To determine where the series converges, we use the Ratio Test:

$$\left| \frac{\dfrac{(-1)^{n+1}}{(2n+3)!} x^{2n+3}}{\dfrac{(-1)^n}{(2n+1)!} x^{2n+1}} \right| = \frac{x^2}{(2n+3)(2n+2)}$$

Taylor and Maclaurin Series

Taylor Series for Some Elementary Functions

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Thus, the series converges for all x, and the radius of convergence is $R = \infty$.

Taylor and Maclaurin Series

Taylor Series for Some Elementary Functions

Example. Find the Maclaurin series for (a) $\sin x$, (b) $\cos x$. Where does each series converge?

Solution. (a)

Remark.
At this stage, as in the previous problem, one still needs to justify that the power series obtained actually represents the function $\sin x$.
We will not address this point here, since it requires tools beyond the scope of this course. Instead, we will establish this fact later using a general theorem.

Taylor and Maclaurin Series

Taylor Series for Some Elementary Functions

Example. Find the Maclaurin series for (a) $\sin x$, (b) $\cos x$. Where does each series converge?

Solution.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1}, \quad \text{for all } x.$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n}, \quad \text{for all } x.$$

Taylor and Maclaurin Series

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Example. Find the Maclaurin series for (a) $\sin x$, (b) $\cos x$. Where does each series converge?

Solution.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \text{for all } x.$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad \text{for all } x.}$$

Homework: Derive the Maclaurin series for $\cos x$.

Taylor and Maclaurin Series

Some Maclaurin Series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1;$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots, \qquad |x| < 1;$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x \le 1;$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad -1 \le x \le 1;$$

Taylor and Maclaurin Series

Other Maclaurin and Taylor Series

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\,, \qquad \text{for all } x;$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\,, \qquad \text{for all } x;$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\,, \qquad \text{for all } x;$$

Taylor and Maclaurin Series

Other Maclaurin and Taylor Series

Example. Obtain Maclaurin series for the following functions:

$$\textbf{(a)}\ e^{-x^2/3}, \quad \textbf{(b)}\ \frac{\sin(x^2)}{x}, \quad \textbf{(c)}\ \sin^2 x.$$

Solution.

Taylor and Maclaurin Series

Other Maclaurin and Taylor Series

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Solution. (a) We substitute $-x^2/3$ for x in the Maclaurin series for e^x:

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Solution. **(a)** We substitute $-x^2/3$ for x in the Maclaurin series for e^x:

$$e^{-x^2/3} = \sum_{n=0}^{\infty} \frac{(-x^2/3)^n}{n!}$$

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Solution. (a) We substitute $-x^2/3$ for x in the Maclaurin series for e^x:

$$\begin{aligned} e^{-x^2/3} &= \sum_{n=0}^{\infty} \frac{(-x^2/3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3^n n!} \\ &= 1 - \frac{x^2}{3 \cdot 1!} + \frac{x^4}{3^2 \cdot 2!} - \frac{x^6}{3^3 \cdot 3!} + \cdots, \quad \text{for all } x. \end{aligned}$$

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(b) If we substitute x^2 for x in the Maclaurin series for $\sin x$ and divide by x, then for $x \neq 0$ we have:

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Note that $f(x) = \sin(x^2)/x$ is undefined at $x = 0$ but does have a limit there since $\lim_{x \to 0} f(x) = 0$. If we define $f(0) = 0$ (the continuous extension of $f(x)$ to $x = 0$), then the series converges to $f(x)$ for all x.

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$$\sin^2 x = \frac{1-\cos(2x)}{2} = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right)$$

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$$= \frac{1}{2}\left(\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \cdots\right)$$

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&= \frac{1}{2}\left(\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \cdots\right) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{(2n+2)!} x^{2n+2} \quad \text{for all } x.
\end{aligned}$$

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Since the series for $\ln(1+t)$ is valid for $-1 < t \le 1$, this series for $\ln x$ is valid for $-1 < (x-2)/2 \le 1$, that is, for $0 < x \le 4$.