MAT123 MATHEMATICS I

Lecture 26: Infinite Sequences and Series (Continued)

Outline

Taylor and Maclaurin Series (Continued)

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$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n}$$

actually hold?

The n**th-degree Taylor polynomial of** f **at** a is

$$\boxed{\begin{aligned} T_n(x) &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}}$$

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actually hold?

Answer.

A function f is equal to the sum of its Taylor series about a if

$$\boxed{\lim_{n\to\infty} T_n(x) = f(x)}$$

for x in an interval around a.

Taylor and Maclaurin Series

$R_n(x) = f(x) - T_n(x)$

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Remainder

Theorem.

If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series of the interval $|x-a| < R$.

Taylor and Maclaurin Series

In trying to show that $\lim_{n \to \infty} R_n(x) = 0$ for a specific function f, we usually use the following fact.

Theorem. *Taylor's Inequality*

If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the Inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d.$$

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Solution. If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n. If d is any positive number and $|x| \leq d$, then $|f^{(n+1)}(x)| = e^x \leq e^d$.

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Solution. If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n. If d is any positive number and $|x| \leq d$, then $|f^{(n+1)}(x)| = e^x \leq e^d$. So Taylor's Inequality, with $a = 0$ and $M = e^d$, says that

$$|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{n+1} \quad \text{for } |x| \leq d.$$

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Notice that the same constant $M = e^d$ works for every value of n.

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$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

because $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

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Therefore $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x. Thus, $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

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Then $|R_n(x)| \to 0$ by the Sandwich Theorem. It follows the $R_n(x) \to 0$ as $n \to \infty$. Therefore, $\sin x$ is equal to the sum of its Maclaurin series for all x.

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f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\
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&\vdots & &\vdots \\
f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots(k-n+1)
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f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots(k-n+1)
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Therefore, the Maclaurin series of $f(x) = (1+x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}\, x^n.$$

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$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right|$$

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$$= \frac{|k-n|}{(n+1)|x|} = \frac{\left|1-\dfrac{k}{n}\right|}{1+\dfrac{1}{n}}|x| \longrightarrow |x| \quad \text{as } n \to \infty.$$

Thus, by the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$.

Taylor and Maclaurin Series

The traditional notation for the coefficients in the binomial series is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

and these numbers are called the **generalized binomial coefficients**.

The Binomial Series

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

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Solution. We write $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}.$$

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Using the binomial series with $k = -\dfrac{1}{2}$ and x replaced by $-\dfrac{x}{4}$, we get

$$\frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^{n}$$

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$$\begin{aligned}
\frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2} &= \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n \\
&= \frac{1}{2}\Bigg[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{x}{4}\right)^3 \\
&\qquad + \cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}\left(-\frac{x}{4}\right)^n\Bigg]
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&= \frac{1}{2}\Bigg[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{x}{4}\right)^3 \\
&\qquad +\cdots+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}\left(-\frac{x}{4}\right)^n\Bigg] \\
&= \frac{1}{2}\left[1+\frac{1}{8}x+\frac{1\cdot 3}{2!8^2}x^2+\frac{1\cdot 3\cdot 5}{3!8^3}x^3+\cdots+\frac{1\cdot 3\cdots(2n-1)}{n!8^n}x^n+\cdots\right]
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Using the binomial series with $k = -\dfrac{1}{2}$ and x replaced by $-\dfrac{x}{4}$, we get

$$\begin{aligned}\frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2} &= \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n \\ &= \frac{1}{2}\left[1+\frac{1}{8}x+\frac{1\cdot 3}{2!8^2}x^2+\frac{1\cdot 3\cdot 5}{3!8^3}x^3+\cdots+\frac{1\cdot 3\cdots(2n-1)}{n!8^n}x^n+\cdots\right]\end{aligned}$$

This series converges when $\left|-\dfrac{x}{4}\right| < 1$, or $|x| < 4$. Thus, the radius of convergence is $R = 4$.

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Integrating term by term, we get

$$\int e^{-x^2}\,dx = \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right) dx$$

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$$\begin{aligned}\int e^{-x^2}\,dx &= \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right) dx \\ &= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots\end{aligned}$$

Taylor and Maclaurin Series

Example. Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

Solution. Replace x by $-x^2$ in the Maclaurin series for e^x:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots .$$

Integrating term by term, we get

$$\begin{aligned} \int e^{-x^2}\,dx &= \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots \right) dx \\ &= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}. \end{aligned}$$

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This series converges for all x because the original series for e^{-x^2} converges for all x.

Taylor and Maclaurin Series

Example. Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

Solution. The Fundamental Theorem of Calculus gives

$$\int_0^1 e^{-x^2}dx = \left[x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \cdots\right]_0^1$$

Taylor and Maclaurin Series

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Taylor and Maclaurin Series

Example. Evaluate $\int e^{-x^2} dx$ as an infinite series.

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$$\begin{aligned} \int_0^1 e^{-x^2} dx &= \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \cdots \right]_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots \\ &\approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475. \end{aligned}$$

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The error involved in this approximation is less than the next term, which is

$$\frac{1}{11\cdot 5!} = \frac{1}{1320} < 0.001.$$

Taylor and Maclaurin Series

Example. (a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) How accurate is this approximation when $7 \leq x \leq 9$?

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Solution.

(a) $f(x) = x^{1/3}$

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Solution.

(a) $f(x) = x^{1/3}$ $\qquad$ $f(8) = 2$

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Solution.

(a) $f(x) = x^{1/3}$ $\qquad f(8) = 2$

$$f'(x) = \frac{1}{3}x^{-2/3} \qquad f'(8) = \frac{1}{12}$$

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Solution.

$$\left.\begin{array}{ll} \text{(a) } f(x) = x^{1/3} & f(8) = 2 \\[6pt] f'(x) = \dfrac{1}{3}x^{-2/3} & f'(8) = \dfrac{1}{12} \\[6pt] f''(x) = -\dfrac{2}{9}x^{-5/3} & f''(8) = -\dfrac{1}{144} \\[6pt] f'''(x) = \dfrac{10}{27}x^{-8/3} & \end{array}\right\}$$

The second-degree Taylor polynomial

$$\begin{aligned} T_2(x) &= f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 \end{aligned}$$

Taylor and Maclaurin Series

Example. (a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.
(b) How accurate is this approximation when $7 \leq x \leq 9$?

Solution.

(a)

$$\left.\begin{aligned} f(x) &= x^{1/3} & f(8) &= 2 \\ f'(x) &= \frac{1}{3}x^{-2/3} & f'(8) &= \frac{1}{12} \\ f''(x) &= -\frac{2}{9}x^{-5/3} & f''(8) &= -\frac{1}{144} \\ f'''(x) &= \frac{10}{27}x^{-8/3} & & \end{aligned}\right\}$$

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The desired approximation is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2.$$

Taylor and Maclaurin Series

Example. (a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.
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Solution. (b) Because $x \geq 7$, we have $x^{8/3} \geq 7^{8/3}$.

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Solution. (b) Because $x \geq 7$, we have $x^{8/3} \geq 7^{8/3}$. So

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \leq \frac{10}{27 \cdot 7^{8/3}} < 0.0021.$$

Taylor and Maclaurin Series

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By Taylor's Inequality, the remainder $R_2(x)$ satisfies

$$|R_2(x)| \leq \frac{M}{3!}|x-8|^3$$

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By Taylor's Inequality, the remainder $R_2(x)$ satisfies

$$|R_2(x)| \leq \frac{M}{3!}|x-8|^3 = \frac{0.0021}{3!} \cdot 1^3$$

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By Taylor's Inequality, the remainder $R_2(x)$ satisfies

$$|R_2(x)| \le \frac{M}{3!}|x-8|^3 = \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6}$$

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By Taylor's Inequality, the remainder $R_2(x)$ satisfies

$$|R_2(x)| \leq \frac{M}{3!}|x-8|^3 = \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004.$$

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By Taylor's Inequality, the remainder $R_2(x)$ satisfies

$$|R_2(x)| \leq \frac{M}{3!}|x-8|^3 = \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004.$$

Thus, if $7 \leq x \leq 9$, the approximation in part (a) is accurate to within 0.0004.