MAT124 MATHEMATICS II

Conics, Parametric Curves, and Polar Curves

Outline

Conics

The **conic sections** (or simply **conics**) are curves obtained by intersecting a plane with a double-napped cone.

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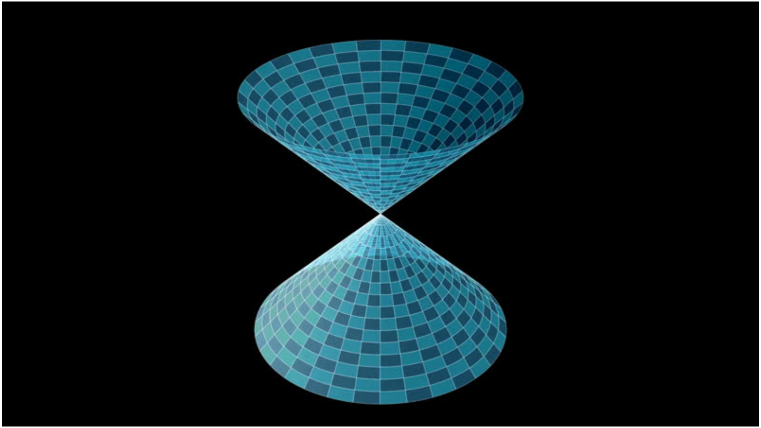
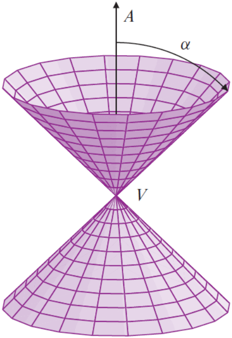
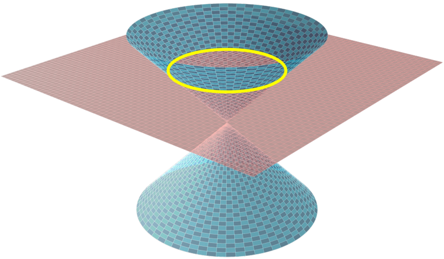


Let α be the semi-vertical angle of the cone, and θ be the angle between the cutting plane and the axis.



Circle



$\theta = 90^\circ$

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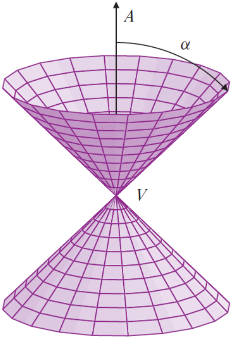
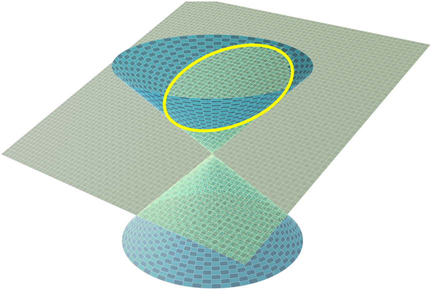


Ellipse



$\alpha < \theta < 90^\circ$

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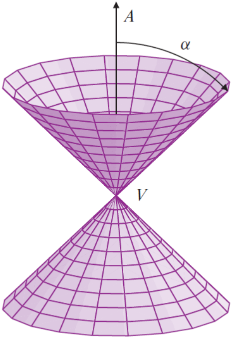
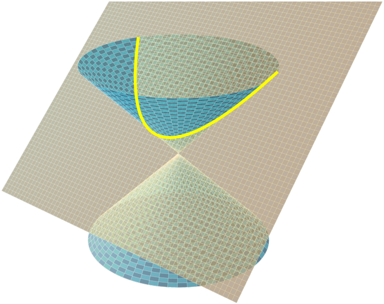


Parabola



$\theta = \alpha$

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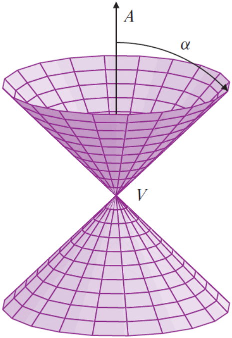
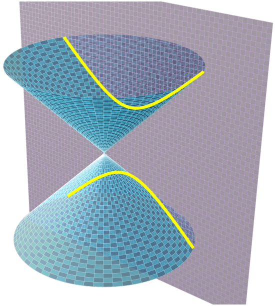


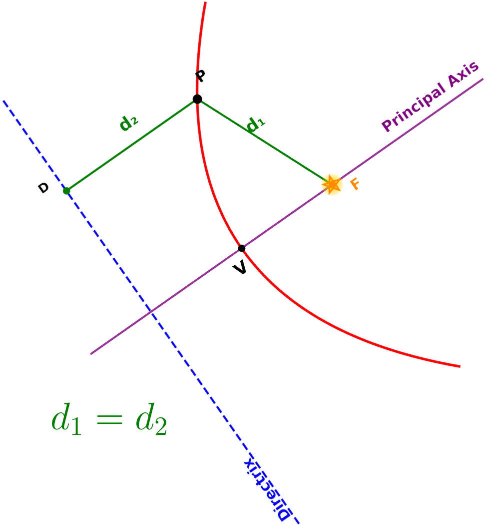
Hyperbola



$0 \leq \theta < \alpha$

Parabolas

A **parabola** is the set of all points in the plane that are equidistant from a fixed point (the **focus**) and a fixed line (the **directrix**).



$$d_1 = d_2$$

Parabolas

Definition

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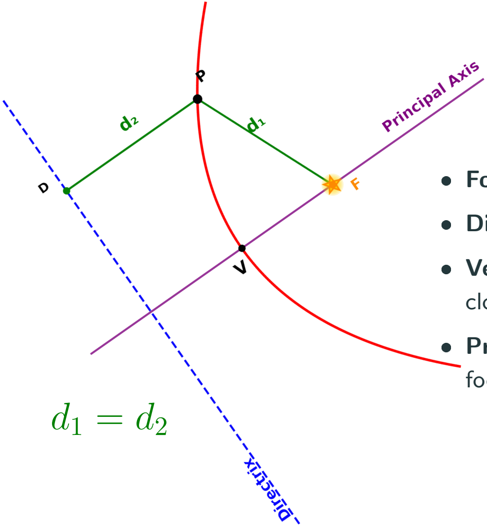

- **Focus** F: The fixed point.
- **Directrix** L: The fixed line.
- **Vertex** V: The point on the parabola closest to the directrix.
- **Principal Axis**: The line through the focus perpendicular to the directrix.

Let the focus be $F = (a, 0)$ and the directrix $x = -a$, where $a > 0$.

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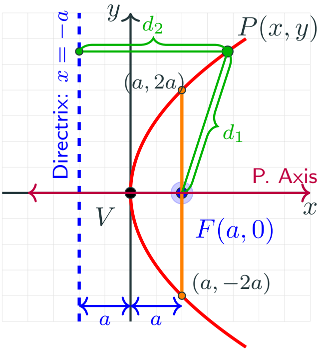
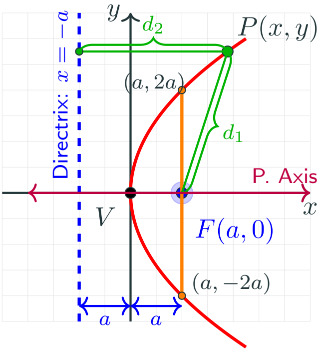


Parabolas

Example

Let the focus be $F = (a, 0)$ and the directrix $x = -a$, where $a > 0$.



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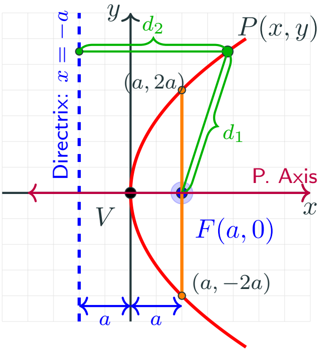


For any point $P = (x, y)$ on the parabola: $d_1 = d_2 \Rightarrow d_1^2 = d_2^2$
Then we have:

$$(x-a)^2 + y^2 = (x+a)^2$$
$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$
$$y^2 = 4ax$$

Parabolas

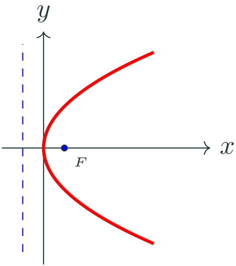
Standard Forms

Equation	Focus	Directrix	Opens
$y^2 = 4ax$	$(a, 0)$	$x = -a$	Right $\rightarrow$
$y^2 = -4ax$	$(-a, 0)$	$x = a$	Left $\leftarrow$
$x^2 = 4ay$	$(0, a)$	$y = -a$	Up $\uparrow$
$x^2 = -4ay$	$(0, -a)$	$y = a$	Down $\downarrow$

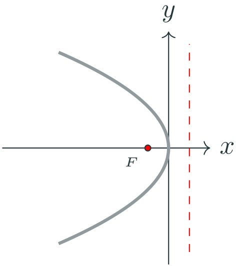
In all cases, $a > 0$ and the vertex is at the origin.

Parabolas

Graphs of Standard Parabolas



$y^2 = 4ax$



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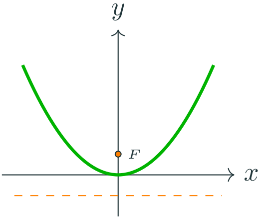
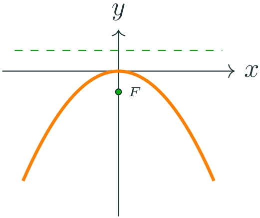


$x^2 = 4ay$



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Parabolas

Shifted Parabolas

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- $(x - h)^2 = 4a(y - k)$ (opens up)
- $(x - h)^2 = -4a(y - k)$ (opens down)

Parabolas

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Parabolas

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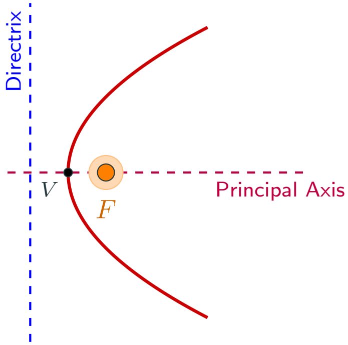
Since the parabola opens upward (focus above vertex):

$$(x - 2)^2 = 4(2)(y - 1) = 8(y - 1)$$

Parabolas

Reflective Property

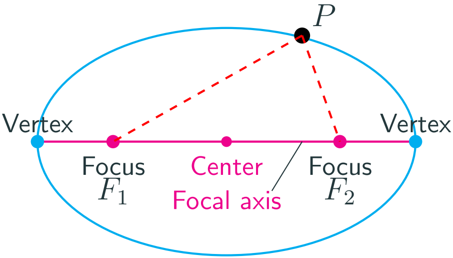
Parabolas have an important **reflective property**: all rays parallel to the axis are reflected through the focus.



Applications: Satellite dishes, reflecting telescopes, solar concentrators, car headlights (reverse).

Ellipses

An **ellipse** is the set of points in a plane such that the sum of the distances from two fixed points (called foci) is constant.



$$|PF_1| + |PF_2| = \text{constant}$$

Example

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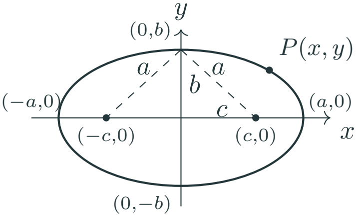

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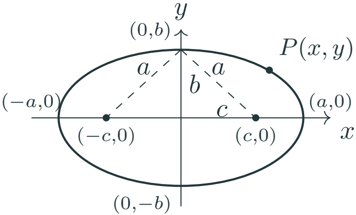

Let $P = (x, y)$ be a point on the ellipse. Then we have:

$$|PF_1|+|PF_2|=\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

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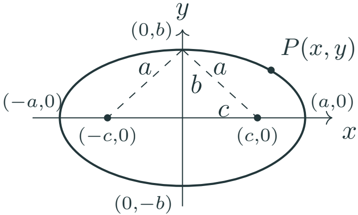

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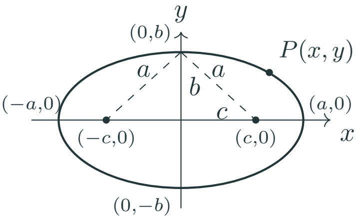

Squaring both sides gives:

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

and so

$$(x-c)^2+y^2=(2a-\sqrt{(x+c)^2+y^2})^2$$

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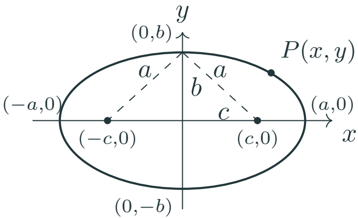

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$$(x-c)^2+y^2=4a^2-4a\sqrt{(x+c)^2+y^2}+(x+c)^2+y^2$$

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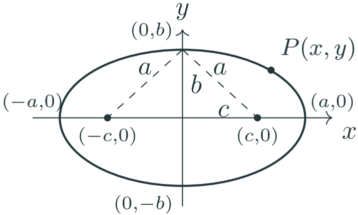

Rearranging this equation gives:

$$a\sqrt{(x+c)^2+y^2}=a^2+cx$$

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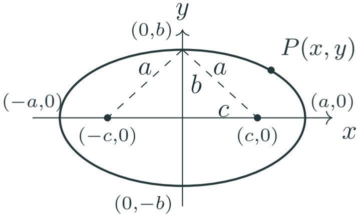


Rearranging this equation gives:

$$a\sqrt{(x+c)^2+y^2}=a^2+cx$$

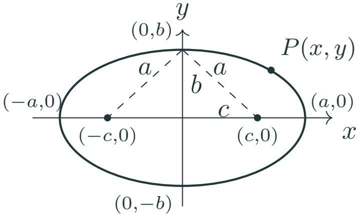
Squaring both sides again yields:

$$a^2((x+c)^2+y^2)=(a^2+cx)^2$$

Example

Find the ellipse with foci at the points $(-c, 0)$ and $(c, 0)$ if the sum of the distances from any point P on the ellipse to these two foci is $2a$ (where $a > c$).

Solution



Now we have:

$$a^2((x+c)^2+y^2) = (a^2+cx)^2$$

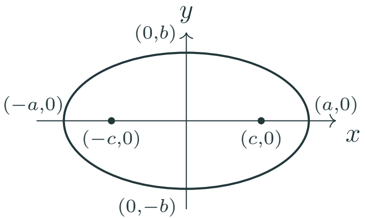
$$a^2(x^2+2cx+c^2+y^2) = a^4+2a^2cx+c^2x^2$$

Simplifying leads to the standard equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

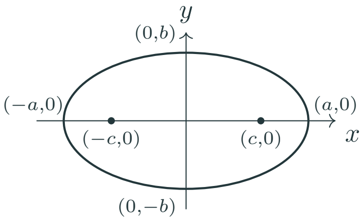
where $b^2=a^2-c^2$.

Ellipses



Standard form of the ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Ellipses



Standard form of the ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The following quantities describe the ellipse:

a	is the **semi-major axis**
b	is the **semi-minor axis**
$c = \sqrt{a^2 - b^2}$	is the **semi-focal separation**

The **eccentricity** of an ellipse is the ratio of the semi-focal separation to the semi-major axis. We denote the eccentricity ε. For the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with $a > b$,

$$\varepsilon = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

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with $a > b$,

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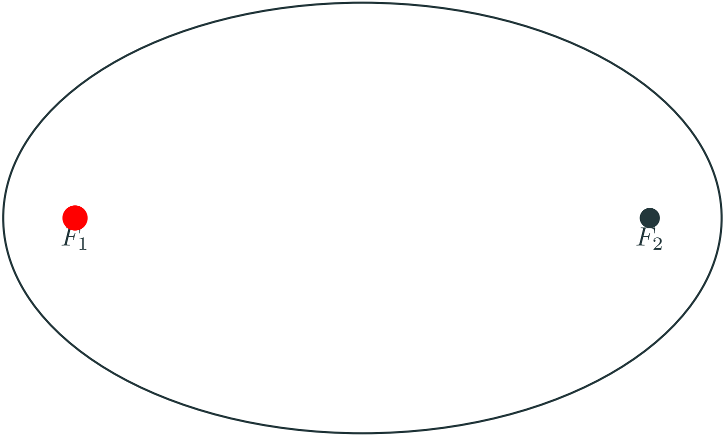
Note that $\varepsilon < 1$ for any ellipse; the greater the value of ε, the more elongated (less circular) is the ellipse. If $\varepsilon = 0$ so that $a = b$ and $c = 0$, the two foci coincide and the ellipse is a circle.

Ellipses

Reflection Property of Ellipses



If $a > b > 0$, each of the lines $x = a/\varepsilon$ and $x = -a/\varepsilon$ is called a **directrix** of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

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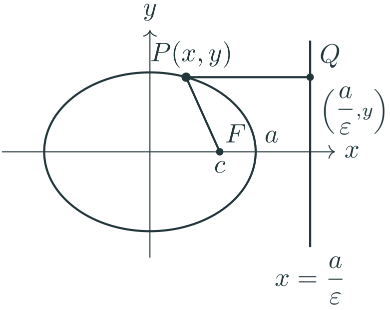

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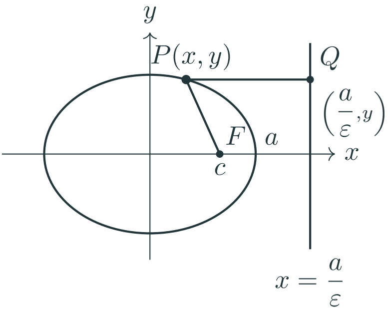

$$
\begin{aligned}
PF^2 &= (x-c)^2 + y^2 \\
&= x^2 - 2cx + c^2 + b^2\left(1 - \frac{x^2}{a^2}\right) \\
&= x^2\left(\frac{a^2 - b^2}{a^2}\right) - 2cx + a^2 - b^2 + b^2 \\
&= \varepsilon^2 x^2 - 2\varepsilon a x + a^2 \qquad \text{(because } c = \varepsilon a\text{)} \\
&= (a - \varepsilon x)^2.
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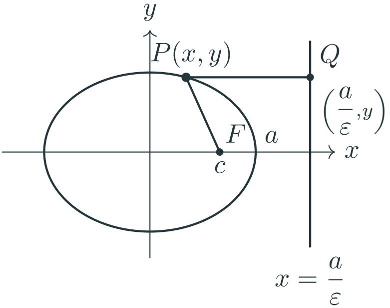

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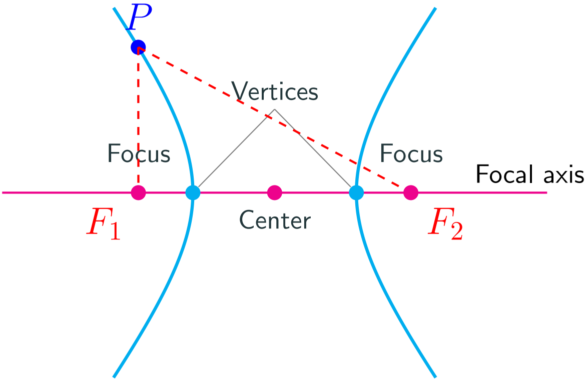
$$

Thus, $PF = a - \varepsilon x$. Also, $QP = (a/\varepsilon - x) = (a - \varepsilon x)/\varepsilon$. Therefore, $PF/QP = \varepsilon$.

Hyperbolas

A **hyperbola** is the set of points in a plane such that the absolute difference of the distances from two fixed points (called foci) is constant.



$$|PF_1 - PF_2| = \text{constant}$$

Hyperbolas

Example

If the foci of a hyperbola are $F_1 = (c, 0)$ and $F_2 = (-c, 0)$, and the difference of the distances from a point $P = (x, y)$ on the hyperbola to the foci is $2a$, then we have:

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Hyperbolas

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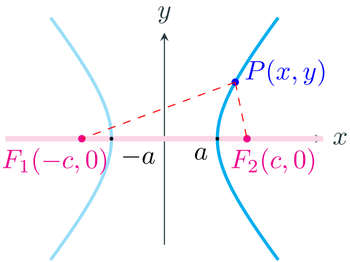
$$PF_2 - PF_1 = \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \begin{cases} 2a & \text{(right branch)} \\ -2a & \text{(left branch).} \end{cases}$$



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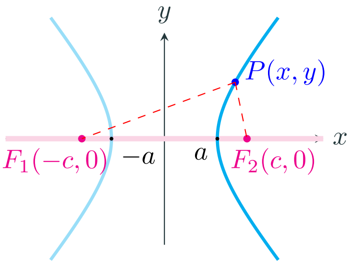
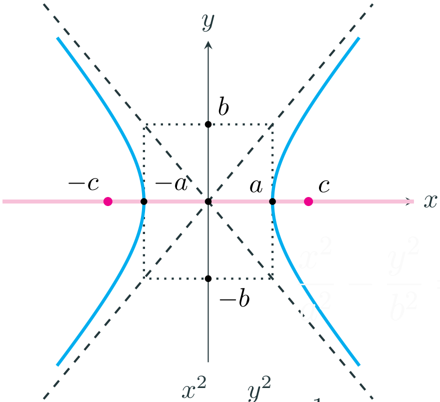


This leads to the standard equation of the hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

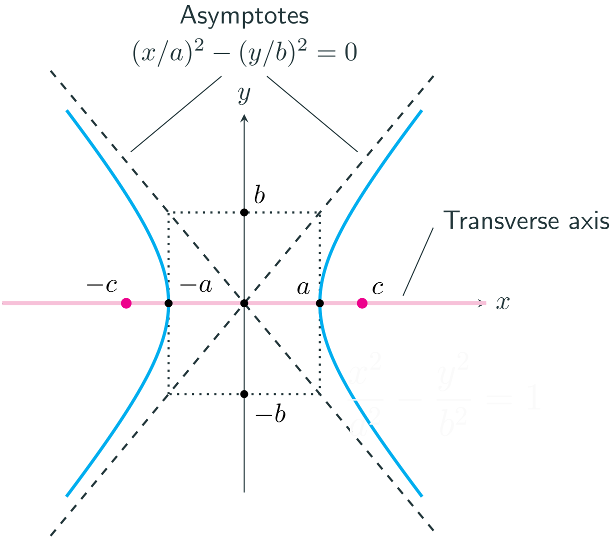
where $b^2 = c^2 - a^2$.



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

a	is the **semi-transverse axis**
b	is the **semi-conjugate axis**
$c = \sqrt{a^2 + b^2}$	is the **semi-focal separation**

Hyperbolas



Asymptotes
$(x/a)^2 - (y/b)^2 = 0$
y
b
Transverse axis
$-c$
$-a$
a
c
x
$-b$
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The eccentricity of the hyperbola is

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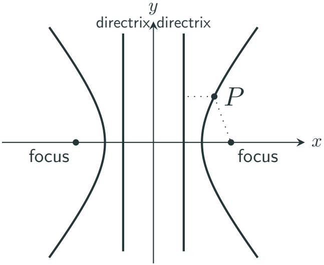
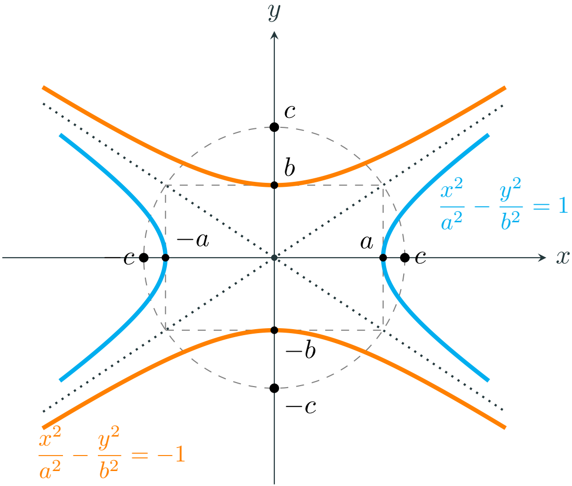


ε is equal to the ratio

$$\frac{\text{distance from } P \text{ to a focus}}{\text{distance from } P \text{ to the corresponding directrix}}$$

Hyperbolas

The two hyperbolas are said to be **conjugate** to each other.



General Conics

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A second-degree equation in two variables,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad (A^2 + B^2 + C^2 > 0),$$

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4. no points at all ($x^2 + y^2 = -1$ is not satisfied by any point in the plane).

General Conics

Example: Describing a Conic Curve

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Solution
We complete the squares in the x and y terms, and rewrite the equation in the form

$$x^2 + 6x + 9 + 2(y^2 - 2y + 1) = 9 + 2 - 7 = 4$$
$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{2} = 1.$$

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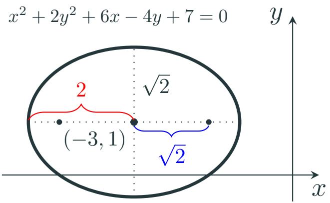
Therefore, it represents an ellipse with:

- **Centre:** $(-3, 1)$
- **Semi-major axis:** $a = 2$
- **Semi-minor axis:** $b = \sqrt{2}$

Since $c = \sqrt{a^2 - b^2} = \sqrt{2}$, the **foci** *are $(-3 \pm \sqrt{2}, 1)$.*



Classifying General Conics: Rotation of Axes

A general second-degree equation with a cross-product term ($B \neq 0$):

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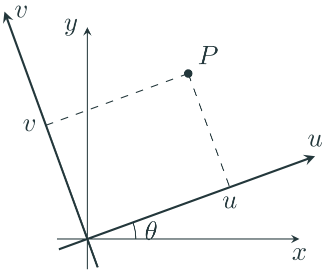
Rotated Equation

$$A'u^2 + B'uv + C'v^2 + D'u + E'v + F = 0$$

$$B' = (C - A)\sin 2\theta + B\cos 2\theta$$



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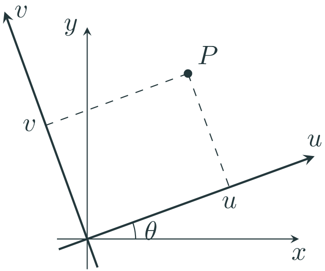
Setting $B' = 0$ gives the rotation angle:

$$\tan 2\theta = \frac{B}{A - C} \quad \text{or} \quad \theta = \frac{\pi}{4} \text{ if } A = C$$



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We have $A = 0$, $B = 1$, *and* $C = 0$. *Since* $A = C$, *we use* $\theta = \pi/4$ *to rotate the axes. The new coordinates are:*

$$u = \frac{x+y}{\sqrt{2}}, \quad v = \frac{-x+y}{\sqrt{2}}$$

$$x = \frac{u-v}{\sqrt{2}}, \quad y = \frac{u+v}{\sqrt{2}}$$

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Identify the curve with equation $xy = 1$.

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$$x = \frac{u-v}{\sqrt{2}}, \quad y = \frac{u+v}{\sqrt{2}}$$

Substituting into the equation gives:

$$xy = \left(\frac{u-v}{\sqrt{2}}\right)\left(\frac{u+v}{\sqrt{2}}\right) = \frac{u^2-v^2}{2} = 1$$

Therefore, the curve is a hyperbola with equation:

$$u^2 - v^2 = 2$$