MAT124 MATHEMATICS II

Parametric Curves

February 18, 2026

Outline

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A **parametric curve** $\mathcal{C}$ in the plane consists of an ordered pair (f, g) of continuous functions each defined on the same interval I. The equations

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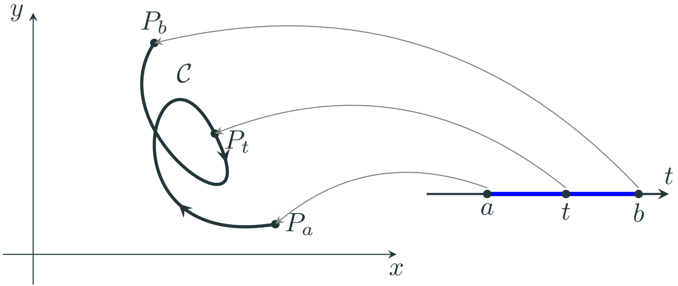

Example: Sketching a Parametric Curve

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Sketch and identify the parametric curve

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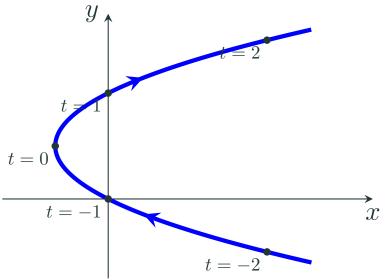
Solution

We use **parameter elimination** *by solving for t in terms of y:*

$$t = y - 1$$
$$x = t^2 - 1 = (y-1)^2 - 1 = y^2 - 2y.$$



Parametric Equations of a Straight Line

Example

The straight line passing through the two points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ has parametric equations:

$$\begin{cases} x = x_0 + t(x_1 - x_0) \\ y = y_0 + t(y_1 - y_0) \end{cases} \qquad (-\infty < t < \infty).$$

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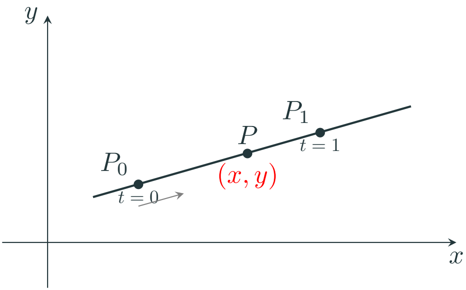

Example: An Arc of a Circle

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$$x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9(1) = 9.$$

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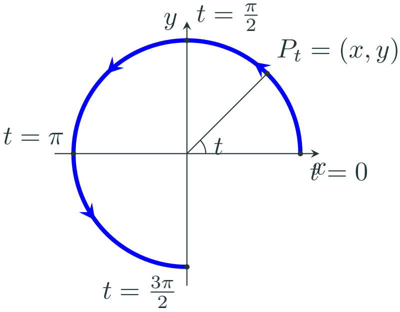

Parametric Equations of an Ellipse

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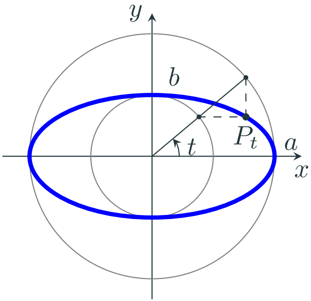
Solution: Dividing by a and b, then squaring and adding:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1.$$

This is an ellipse centered at the origin with semi-axes a and b.



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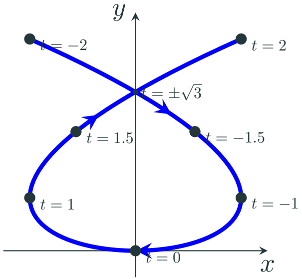

t	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
x	-2	$\frac{9}{8}$	2	$\frac{11}{8}$	0	$-\frac{11}{8}$	-2	$-\frac{9}{8}$	2
y	4	$\frac{9}{4}$	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	$\frac{9}{4}$	4

$\rightarrow$ symmetric about the y-axis because x is an odd function of t and y is an even function of t.

Plane curves

A **plane curve** is a set of points (x, y) in the plane such that $x = f(t)$ and $y = g(t)$ for some t in an interval I, where f and g are continuous functions defined on I. Any such interval I and function pair (f, g) that generate the points of $\mathcal{C}$ is called a **parametrization** of $\mathcal{C}$.

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Since a plane curve does not involve any specific parametrization, it has no specific direction.

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EXAMPLE

If f is a continuous function defined on an interval I, then the plane curve defined by $x = t$ and $y = f(t)$ for $t \in I$ is the graph of the function f.

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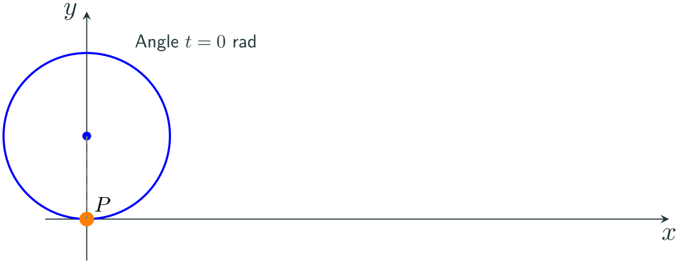
If a circle rolls without slipping along a straight line, find the path followed by a point fixed on the circle.



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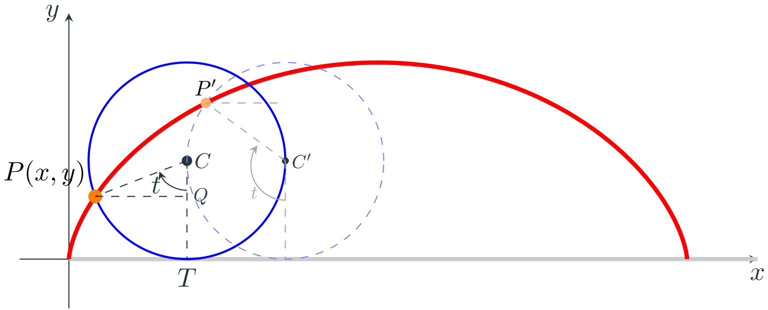

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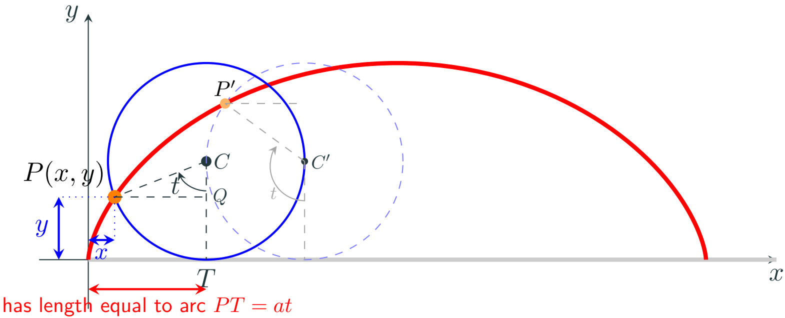

$$x = OT - PQ = at - a\sin(\pi - t) = at - a\sin t$$

$$y = TC + CQ = a + a\cos(\pi - t) = a - a\cos t$$

Result: $x = a(t - \sin t), \quad y = a(1 - \cos t)$

Definition: Smoothness

We say that a plane curve is **smooth** if it has a tangent line at each point P and this tangent turns in a continuous way as P moves along the curve. In other words, the angle between the tangent line at P and some fixed line (say, the x-axis) is a continuous function of the position of P.

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A curve $y = f(x)$ is smooth on any interval where $f'(x)$ exists and is continuous.

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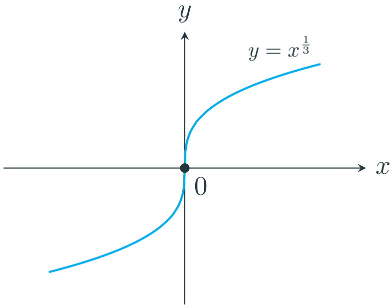
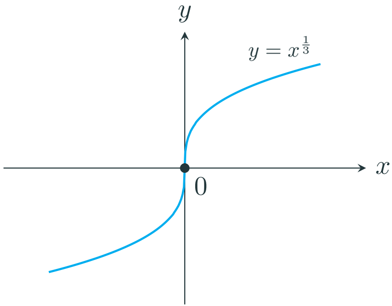


A curve may be smooth on an interval containing **isolated singularities**.

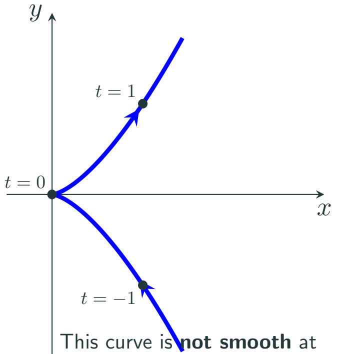
Consider the curve: $y = x^{1/3}$

At the origin ($x = 0$), the tangent line exists (it is the vertical y-axis), and the curve is considered smooth there despite the vertical slope.



Smooth Parametric Curves

Smooth Parametric Curves and Their Slopes



This curve is **not smooth** at the origin but has a **cusp** there.

Parametric Equations:

$$x = t^2, \quad y = t^3$$

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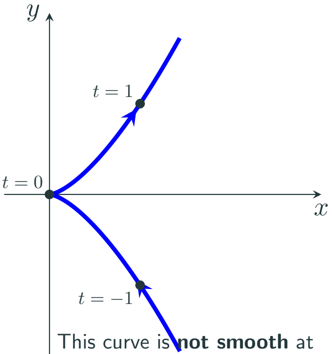

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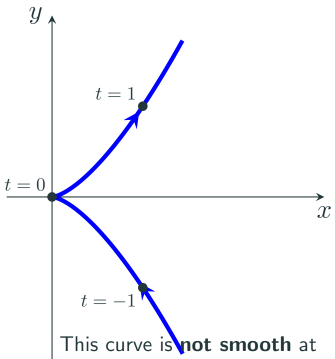

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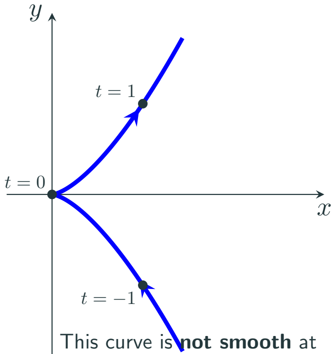
Differentiation:

$$\frac{dx}{dy} = \frac{2}{3}y^{-1/3}$$

As $y \to 0^{\pm}$:

$$\lim_{y\to 0^{\pm}} \frac{dx}{dy} = \pm\infty$$



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Theorem: The Slope of a Parametric Curve
Let $\mathcal{C}$ be the parametric curve $x = f(t)$, $y = g(t)$, where $f'(t)$ and $g'(t)$ are continuous on an interval I.

If $f'(t) \neq 0$ on I, then $\mathcal{C}$ is smooth and has at each t a tangent line with slope:

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Important Note
Thus, $\mathcal{C}$ is smooth except possibly at points where $f'(t)$ and $g'(t)$ are both 0.

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If f' and g' are continuous, and both vanish at some point t_0, then the curve $x = f(t)$, $y = g(t)$ **may or may not** be smooth around t_0.

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$$x = t^3, \quad y = t^6$$

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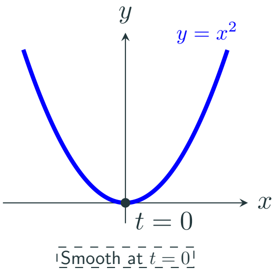
- This is just the parabola $y = x^2$, which is **smooth everywhere**.



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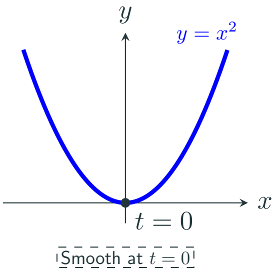
- Notice that:

$$\frac{dx}{dt} = 3t^2, \quad \frac{dy}{dt} = 6t^5$$

Both derivatives **vanish at** $t = 0$.



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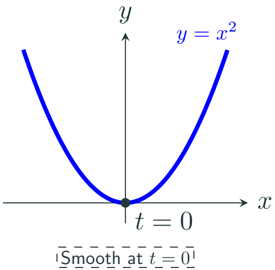

This illustrates that the vanishing of derivatives doesn't always imply a lack of smoothness.

Smooth Parametric Curves and Their Slopes

Tangents and Normals to Parametric Curves

If f' and g' are continuous and not both 0 at t_0, then the parametric equations:

$$\begin{cases} x = f(t_0) + f'(t_0)(t - t_0) \\ y = g(t_0) + g'(t_0)(t - t_0) \end{cases} \qquad (-\infty < t < \infty)$$

represent the **tangent line** to the parametric curve $x = f(t)$, $y = g(t)$ at the point $(f(t_0), g(t_0))$.

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The **normal line** there has parametric equations:

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Both lines pass through $(f(t_0), g(t_0))$ when $t = t_0$.

EXAMPLE:
Find equations of the tangent and normal lines to the parametric curve $x = t^2 - t,\ y = t^2 + t$ at the point where $t = 2$.

Solution: At $t = 2$, we have $x = 2$ and $y = 6$.

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Hence, the tangent and the normal lines have parametric equations:

Tangent: $\begin{cases} x = 2 + 3(t-2) = 3t - 4 \\ y = 6 + 5(t-2) = 5t - 4 \end{cases}$

Normal: $\begin{cases} x = 2 + 5(t-2) = 5t - 8 \\ y = 6 - 3(t-2) = -3t + 12 \end{cases}$

The concavity of a parametric curve can be determined using the second derivatives of the parametric equations. The procedure is just to calculate d^2y/dx^2 using the **Chain Rule**:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\frac{g'(t)}{f'(t)} = \frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)\frac{dt}{dx} \\ &= \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^2} \cdot \frac{1}{f'(t)}\end{aligned}$$

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$$= \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^2} \cdot \frac{1}{f'(t)}$$

On an interval where $f'(t) \neq 0$, the parametric curve $x = f(t)$, $y = g(t)$ has concavity determined by

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$$f'(t) = 3(t^2 - 1) = 3(t-1)(t+1), \quad g'(t) = 2t$$

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$$f'(t) = 3(t^2 - 1) = 3(t-1)(t+1), \quad g'(t) = 2t$$

t	-2		-1		0		1		2
$f'(t)$		$+$	0	$-$	$-$	$-$	0	$+$	
$g'(t)$		$-$	$-$	$-$	0	$+$	$+$	$+$	
x		$\rightarrow$	$\cdot$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\cdot$	$\rightarrow$	
y		$\downarrow$	$\downarrow$	$\downarrow$	$\cdot$	$\uparrow$	$\uparrow$	$\uparrow$	
curve		$\searrow$	$\downarrow$	$\swarrow$	$\leftarrow$	$\nwarrow$	$\uparrow$	$\nearrow$	

Sketching Parametric Curves

EXAMPLE:
Use slope and concavity information to sketch the graph of:

$$x = f(t) = t^3 - 3t, \quad y = g(t) = t^2, \quad (-2 \le t \le 2)$$

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x		$\rightarrow$	$\cdot$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\cdot$	$\rightarrow$	
y		$\downarrow$	$\downarrow$	$\downarrow$	$\cdot$	$\uparrow$	$\uparrow$	$\uparrow$	
curve		$\searrow$	$\downarrow$	$\swarrow$	$\leftarrow$	$\nwarrow$	$\uparrow$	$\nearrow$	

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3} = \frac{6(t^2-1) - 12t^2}{27(t^2-1)^3} = \frac{-2(t^2+1)}{9(t^2-1)^3}$$

The curve is **concave down** on $(-2, -1) \cup (1, 2)$ and **concave up** on $(-1, 1)$.

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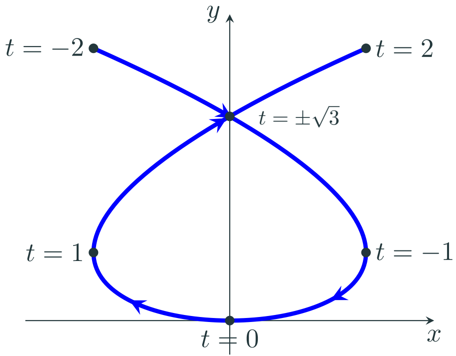
Solution: ... and the sketch is as follows:



Symmetric about y-axis

Arc Length of a Parametric Curve

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The **length** of the curve $\mathcal{C}$ is given by

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Squaring these formulas, adding and simplifying:

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 \\ &= e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2\sin t \cos t \\ &\quad + \cos^2 t) \\ &= e^{2t}(2) = 2e^{2t}.\end{aligned}$$

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The length of the curve is, therefore:

$$s = \int_0^2 \sqrt{2e^{2t}}\, dt = \sqrt{2}\int_0^2 e^t\, dt = \sqrt{2}(e^2 - 1) \text{ units.}$$

Surface Area of Revolution for Parametric Curves

Surface Area of Revolution for Parametric Curves

If the smooth curve $x = f(t),\ y = g(t)\ (a \leq t \leq b)$ is rotated about the coordinate axes, the area S of the generated surface is given by:

1. Rotation about the x-axis:

$$S = 2\pi \int_a^b \underbrace{|g(t)|}_{|y|} \underbrace{\sqrt{(f'(t))^2 + (g'(t))^2}\,dt}_{ds}$$

2. Rotation about the y-axis:

$$S = 2\pi \int_a^b \underbrace{|f(t)|}_{|x|} \underbrace{\sqrt{(f'(t))^2 + (g'(t))^2}\,dt}_{ds}$$

Note: The term ds represents the differential arc length element derived in the previous section.

Surface Area of Revolution for Parametric Curves

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Find the area of the surface of revolution obtained by rotating the astroid $x = a\cos^3 t,\ y = a\sin^3 t\ (a > 0)$ about the x-axis.

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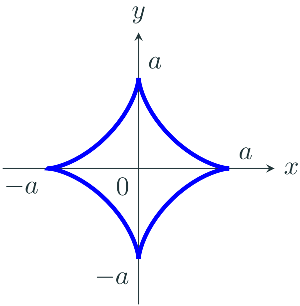

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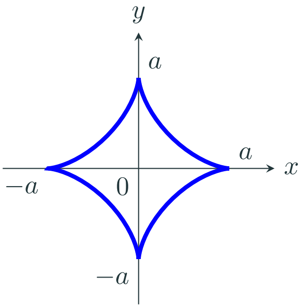

$$\frac{dx}{dt} = -3a\cos^2 t \sin t$$
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Using symmetry (upper half):

$$ds = \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}\,dt$$
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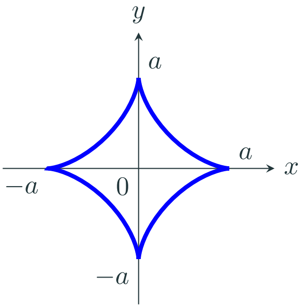

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The surface area S is:

$$\begin{aligned} S &= 2\times 2\pi\int_0^{\pi/2} a\sin^3 t(3a\cos t\sin t)\,dt \\ &= 12\pi a^2\int_0^{\pi/2}\sin^4 t\cos t\,dt \end{aligned}$$

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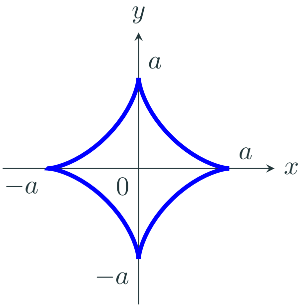

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Let $u = \sin t,\ du = \cos t\,dt$:

$$S = 12\pi a^2\int_0^1 u^4\,du = \frac{12\pi a^2}{5}$$ sq. units.

Areas Bounded by Parametric Curves

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Consider a parametric curve $\mathcal{C}$ with equations $x = f(t),\ y = g(t)\ (a \leq t \leq b)$, where f is differentiable and g is continuous on $[a, b]$.

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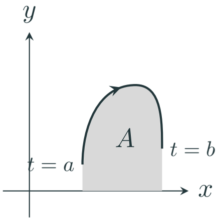
Consider a parametric curve $\mathcal{C}$ with equations $x = f(t),\ y = g(t)\ (a \le t \le b)$, where f is differentiable and g is continuous on $[a, b]$.

If $\mathcal{C}$ is a non-self-intersecting closed curve, the area A of the region bounded by $\mathcal{C}$ is given by:

Area Formulas

- $A = \int_a^b g(t)f'(t)\,dt$ if $\mathcal{C}$ is traversed **clockwise** as t increases.
- $A = -\int_a^b g(t)f'(t)\,dt$ if $\mathcal{C}$ is traversed **counterclockwise**.



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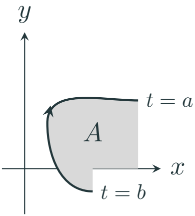

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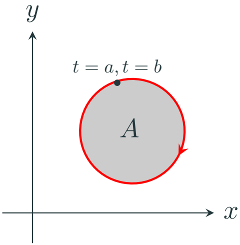
Areas Bounded by Closed Parametric Curves

If the parametric curve $\mathcal{C}$ is **closed** (i.e., the point for $t = a$ and $t = b$ is the same), the area A of the enclosed region is determined by the direction of traversal.



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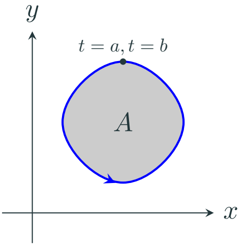

$A = -\int_a^b g(t) f'(t)\, dt$

Note: Clockwise (Red) results in a positive integral, Counter-clockwise (Blue) requires a negative sign for area.

Areas Bounded by Closed Parametric Curves

Problem

Find the area bounded by the ellipse $x = a\cos s,\ y = b\sin s$, $(0 \le s \le 2\pi)$.

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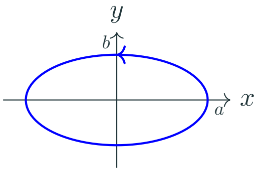
Solution: This ellipse is traversed **counterclockwise** as s increases from 0 to 2π. Therefore, we use the negative integral formula:

$$\begin{aligned} A &= -\int_0^{2\pi} g(s)f'(s)\,ds \\ &= -\int_0^{2\pi} (b\sin s)(-a\sin s)\,ds \end{aligned}$$



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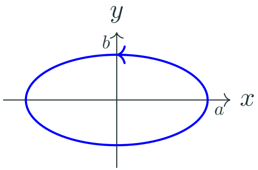
\begin{aligned}
A &= -\int_0^{2\pi} g(s)f'(s)\,ds \\
&= -\int_0^{2\pi} (b\sin s)(-a\sin s)\,ds \\
&= ab\int_0^{2\pi} \sin^2 s\,ds \\
&= \frac{ab}{2}\int_0^{2\pi} (1-\cos 2s)\,ds
\end{aligned}
$$



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&= \left[\frac{ab}{2}s - \frac{ab}{4}\sin 2s\right]_0^{2\pi} = \pi ab \text{ square units.}
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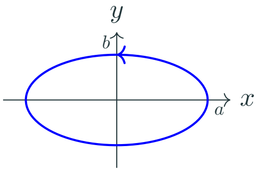

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Problem

Find the area above the x-axis and under one arch of the cycloid:

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

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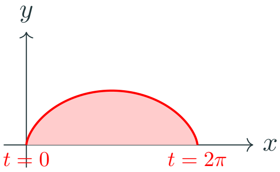
Solution: One arch is generated as t goes from 0 to 2π. Since $y \geq 0$ and $dx/dt = a(1 - \cos t) \geq 0$, the area is:



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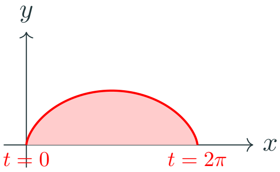

$$A = \int_0^{2\pi} y\,dx = \int_0^{2\pi} y(t)\frac{dx}{dt}\,dt = \int_0^{2\pi} a(1 - \cos t) \cdot a(1 - \cos t)\,dt$$

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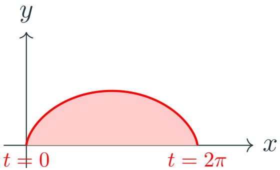

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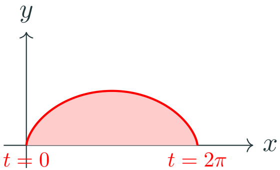

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$$= a^2 \int_0^{2\pi} (1 - \cos t)^2\, dt = a^2 \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2}\right) dt$$

$$= a^2 \left[t - 2\sin t + \frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} = 3\pi a^2 \text{ sq. units.}$$