MAT124 MATHEMATICS II

Polar Curves

Outline

Circles in Polar Coordinates

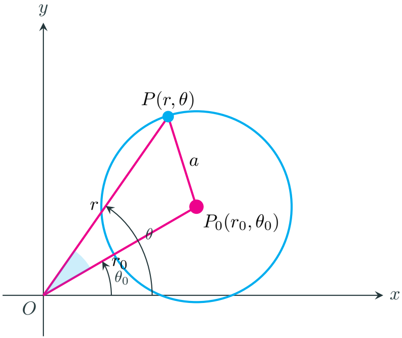
Slopes for Polar Curves

Areas in Polar Coordinates

Length of a Polar Curve

Circles in Polar Coordinates

The General Polar Equation for Circles



Circles in Polar Coordinates

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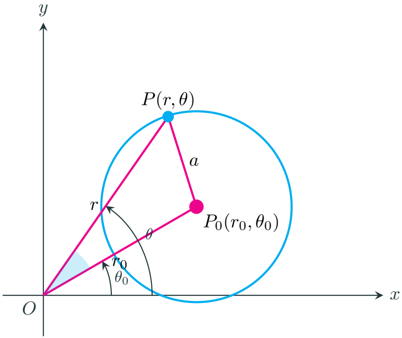

Applying the **Law of Cosines** to the triangle OPP_0:

Standard Equation

$$a^2 = r_0^2 + r^2 - 2r_0 r\cos(\theta - \theta_0)$$

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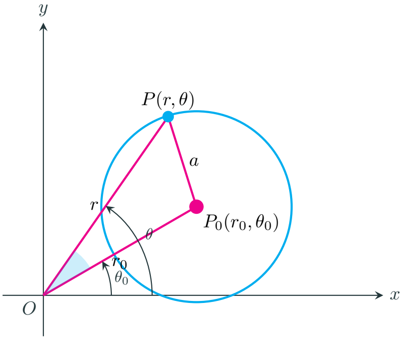

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Circles in Polar Coordinates

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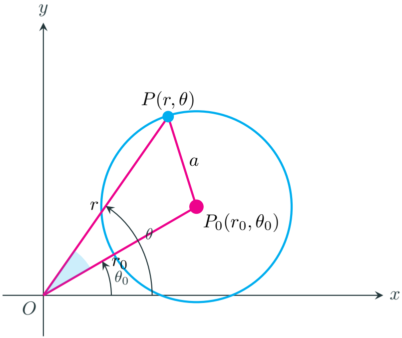

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- Special cases (like the circle passing through the origin) can be derived by setting $r_0 = a$.

Circles in Polar Coordinates

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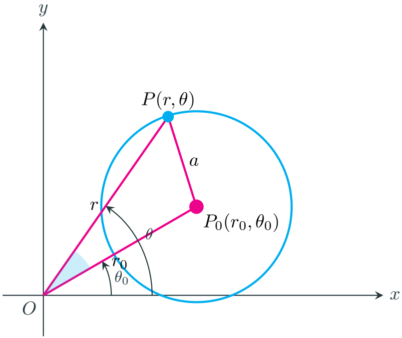

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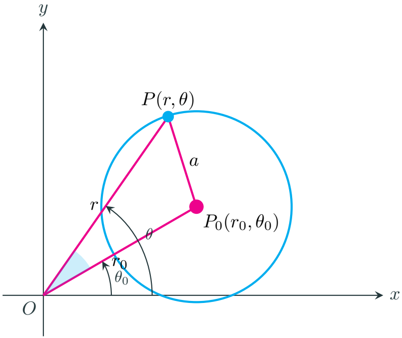

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- If the center lies on the positive y-axis, $\theta_0 = \pi/2$, and we get

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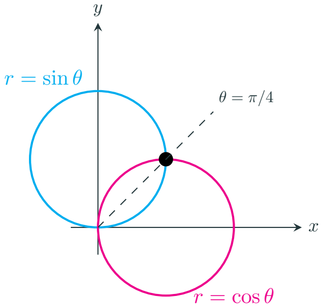

For $r = \sin\theta$ and $r = \cos\theta$:

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- Point $(1/\sqrt{2}, \pi/4)$ is found.

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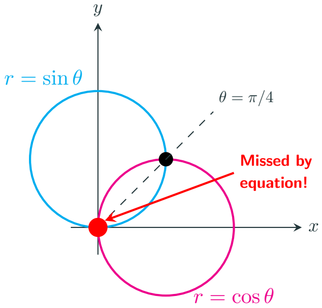

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Solution: Since both functions of θ are periodic with period 2π, we need only look for solutions satisfying $0 \le \theta \le 2\pi$:

$$\sin\theta = 1 - \sin\theta \implies 2\sin\theta = 1 \implies \sin\theta = 1/2 \implies$$

$$\theta = \pi/6 \quad \text{or} \quad \theta = 5\pi/6$$

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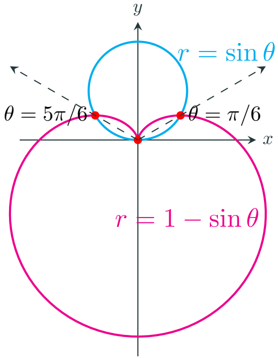

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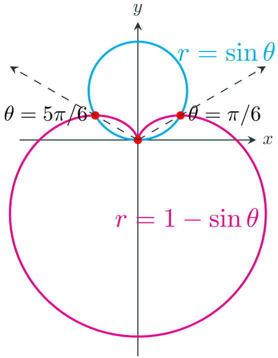

Slopes for Polar Curves

Let $r = f(\theta)$ be a differentiable function of θ. To find the slope of the tangent line, we treat the polar curve as a set of parametric equations:

Parametric Equations

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta$$

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Tangent Lines to Polar Curves

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If $dx/d\theta \neq 0$, the slope dy/dx is given by the parametric formula:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

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- **Vertical Tangents:** Occur when $dx/d\theta = 0$ (provided $dy/d\theta \neq 0$).

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Solution: Using parametric relations $y = r\sin\theta = (1 + \cos\theta)\sin\theta$:

For horizontal tangents, we set $dy/d\theta = 0$:

$$\begin{aligned} 0 = \frac{dy}{d\theta} &= -\sin^2\theta + \cos^2\theta + \cos\theta \\ &= 2\cos^2\theta + \cos\theta - 1 \\ &= (2\cos\theta - 1)(\cos\theta + 1) \end{aligned}$$

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- **Conclusion:** Horizontal tangents are at $[3/2, \pm\pi/3]$. At $\theta = \pi$, we have $r = 0$ (the cusp).

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Solution (continued): For the cardioid $r = 1 + \cos\theta$, vertical tangents occur when $dx/d\theta = 0$:

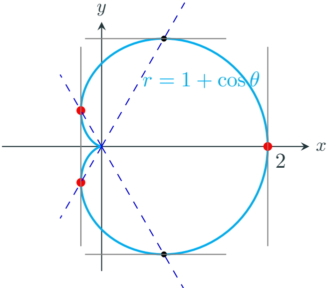
$$0 = \frac{dx}{d\theta} = -\sin\theta - 2\cos\theta\sin\theta$$
$$= -\sin\theta(1 + 2\cos\theta)$$

The solutions are:

- $\sin\theta = 0 \implies \theta = 0, \pi$
- $\cos\theta = -1/2 \implies \theta = \pm 2\pi/3$

There are **vertical tangent lines** at:

$$[2, 0] \text{ and } [1/2, \pm 2\pi/3]$$



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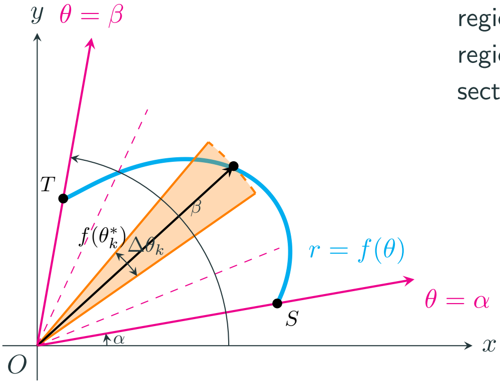

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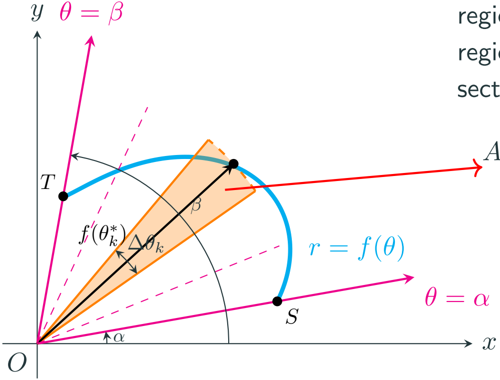

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$$A_k = \frac{1}{2} r_k^2 \Delta\theta_k = \frac{1}{2}(f(\theta_k^*))^2 \Delta\theta_k$$

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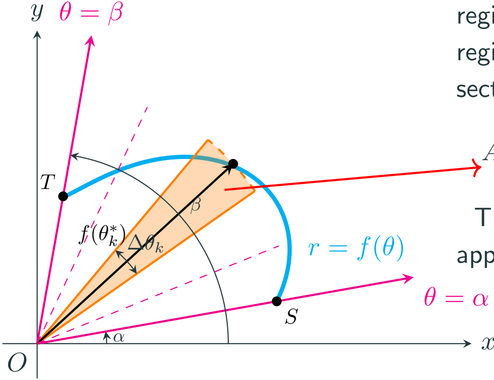

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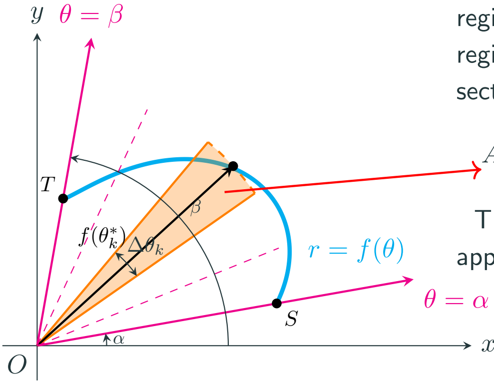

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$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k^*)\right)^2 \Delta\theta_k = \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 d\theta$$

Areas in Polar Coordinates

Area of the Fan-Shaped Region

The area of the region bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$ is given by the following integral:

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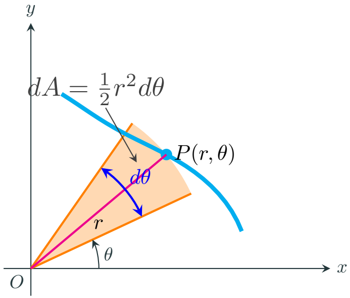
This is the integral of the **area differential**:

$$dA = \frac{1}{2} r^2 \, d\theta$$

Geometrically, dA represents the area of an infinitesimal circular sector with radius r and central angle $d\theta$.



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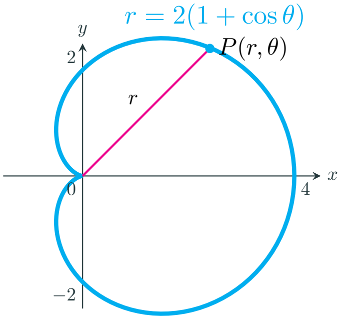

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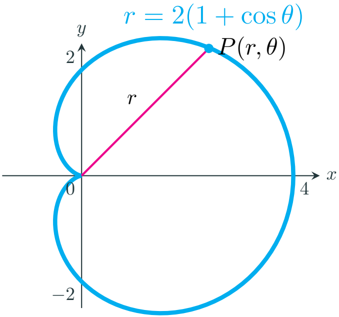

The area is given by the integral from $\theta = 0$ to 2π:

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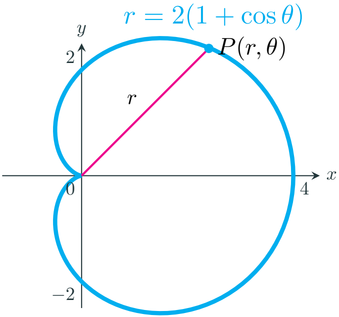

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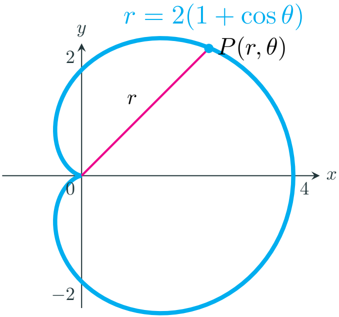

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$$= \int_0^{2\pi} \left(2 + 4\cos\theta + 2 \cdot \frac{1 + \cos 2\theta}{2}\right) d\theta$$

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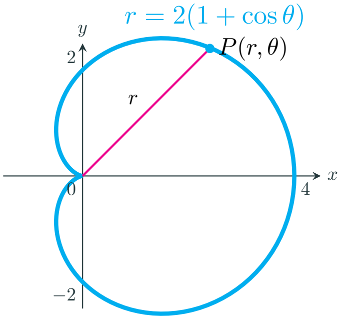

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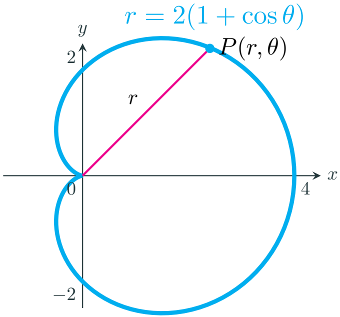

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&= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta)\, d\theta \\
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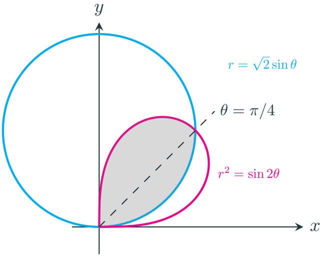

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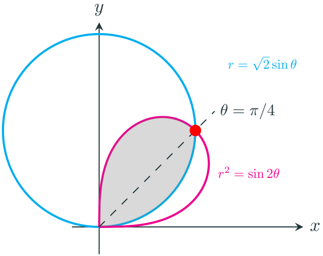

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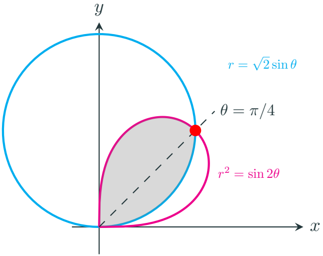

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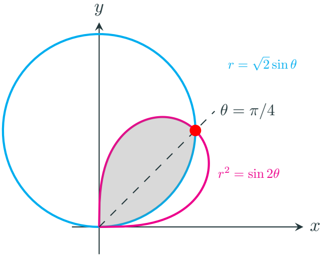

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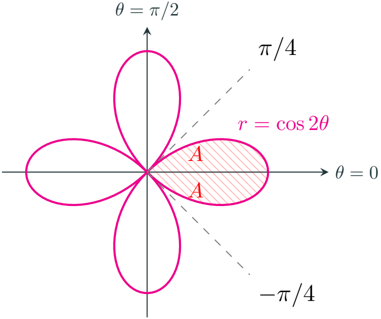

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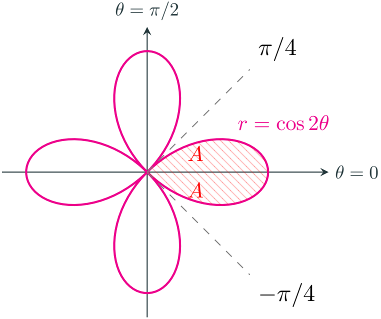

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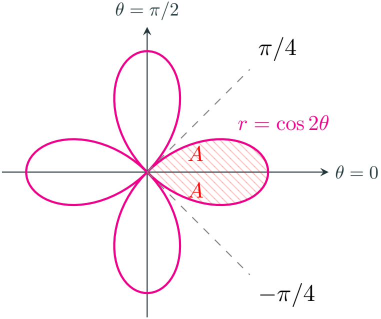

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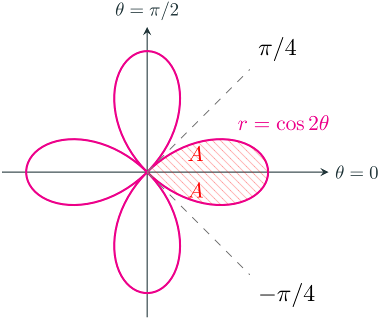

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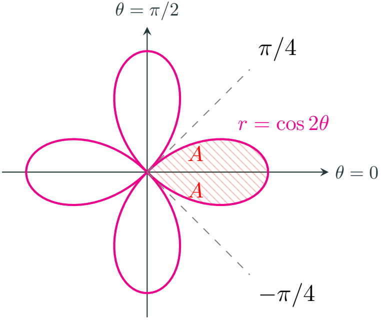

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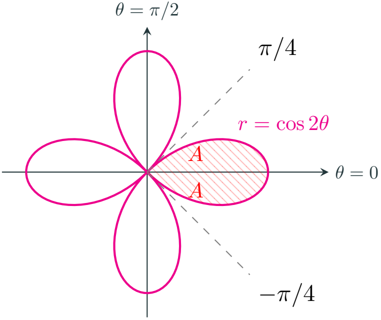

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$$= \frac{\pi}{8} \text{ square units}$$

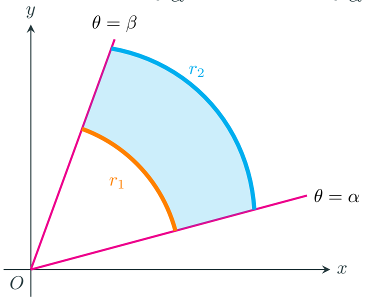
Area Between Polar Curves

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \alpha \le \theta \le \beta$

The area A of the region bounded by two polar curves and two rays is given by the difference of the areas enclosed by each curve:

General Area Formula

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2 \, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 \, d\theta = \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2) \, d\theta$$



- r_2 is the outer boundary curve.
- r_1 is the inner boundary curve.
- The area is obtained by integrating the difference of the squares of the radii.

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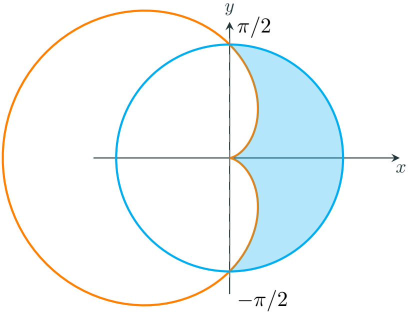

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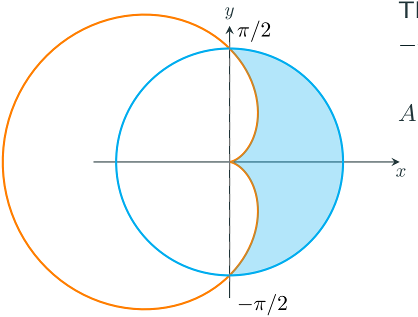

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The area A is found by integrating from $-\pi/2$ to $\pi/2$:

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(r_2^2 - r_1^2)\, d\theta = 2\int_{0}^{\pi/2} \frac{1}{2}(r_2^2 - r_1^2)\, d\theta$$

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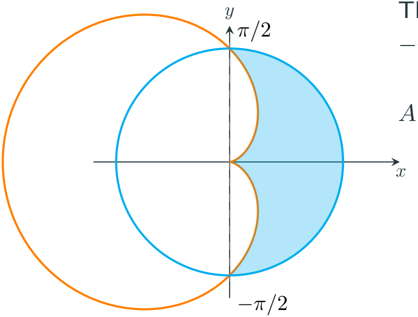

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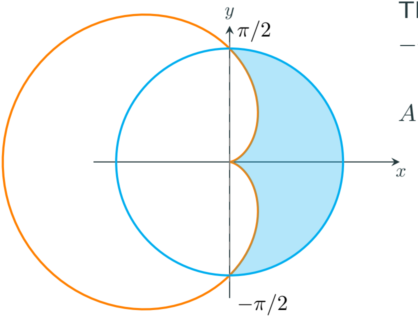

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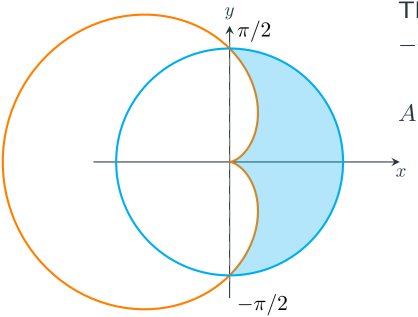

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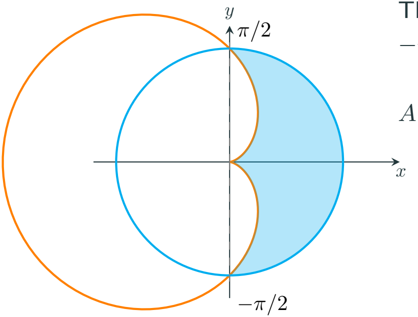

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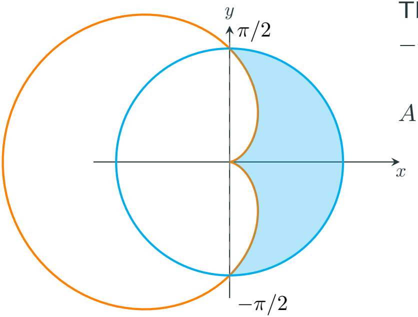

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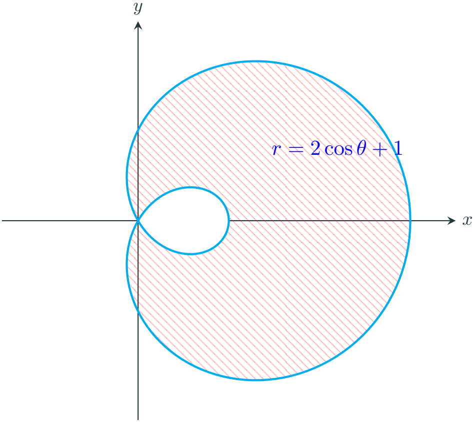
$$= 2 - \frac{\pi}{4} \text{ square units}$$

EXERCISE: Find the area of the shaded region below for the limacon curve $r = 2\cos\theta + 1$.



Length of a Polar Curve

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If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve exactly once as θ runs from α to β, then the length of the curve is given by:

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Polar Arc Length Formula

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

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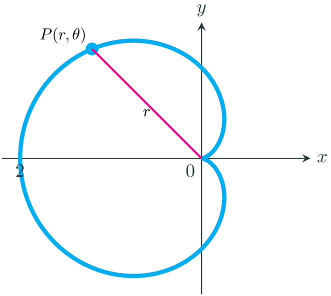

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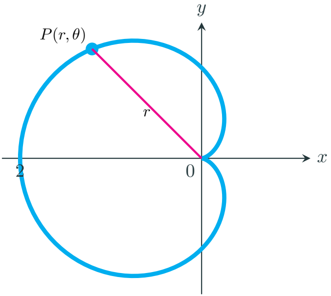

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - \cos\theta)^2 + \sin^2\theta = 2 - 2\cos\theta$$

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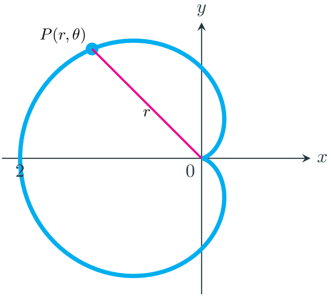

$r^2 + (\frac{dr}{d\theta})^2 = (1 - \cos\theta)^2 + \sin^2\theta = 2 - 2\cos\theta$

Using $1 - \cos\theta = 2\sin^2(\theta/2)$:

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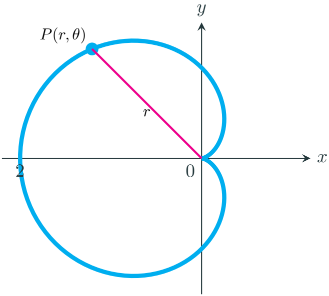

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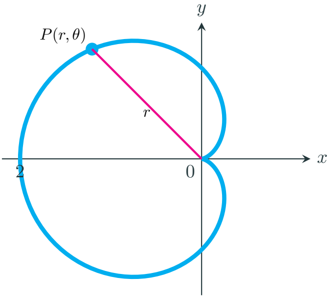

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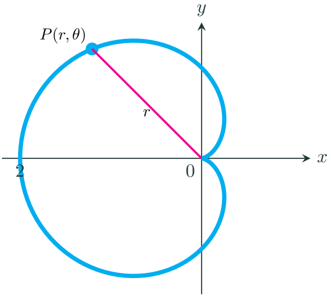

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Since $\sin(\theta/2) \geq 0$ for $0 \leq \theta \leq 2\pi$:

$$L = \int_0^{2\pi} 2\sin(\theta/2)\, d\theta = [-4\cos(\theta/2)]_0^{2\pi} = (-4\cos\pi) - (-4\cos 0)$$

$= 4 + 4 = 8$ units