MAT124 MATHEMATICS II

Analytic Geometry in 3-Dimensional Space and Vectors

Outline

Analytic Geometry in 3-Dimensional Space

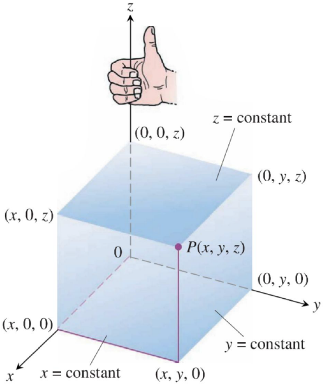
- The Cartesian Coordinate System
- Euclidean n-Space
- Vectors in 2- and 3-Space

Analytic Geometry in 3-Dimensional Space

Analytic Geometry in Three Dimensions

The Cartesian Coordinate System

The three-dimensional coordinate system is formed by three mutually perpendicular axes meeting at the **origin** $(0, 0, 0)$.



Right-Handed System

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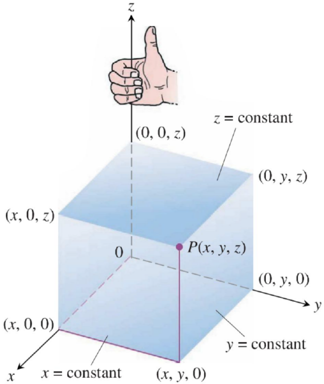

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The coordinate axes determine three fundamental planes:

xy-plane: Equation is $z = 0$.

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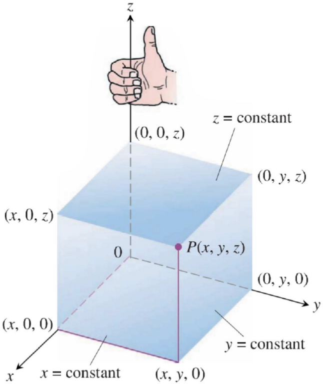

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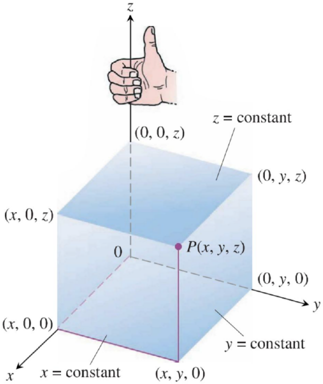

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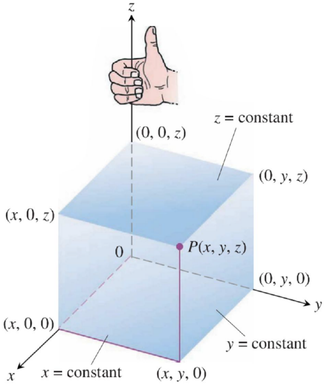

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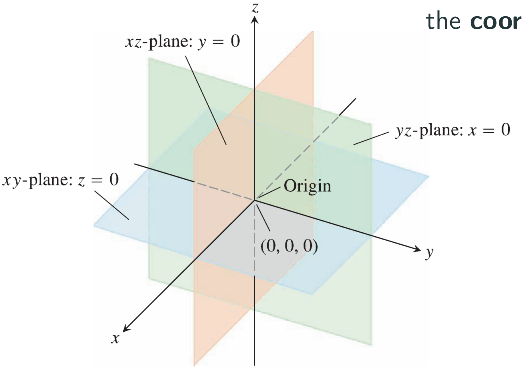
Octants

These planes divide space into eight regions called **octants**. The first octant is where $x, y, z > 0$.

Analytic Geometry in Three Dimensions



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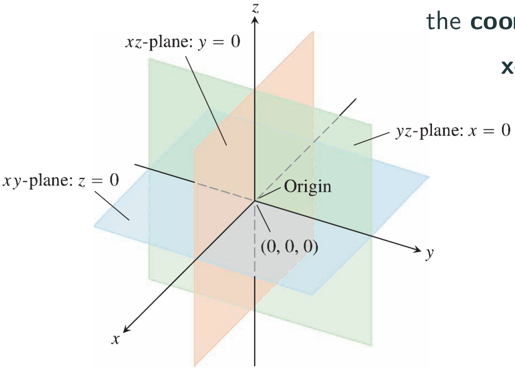

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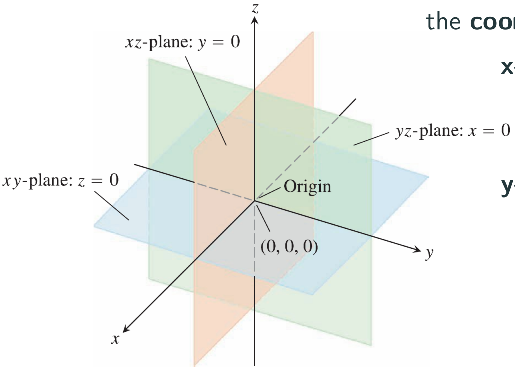

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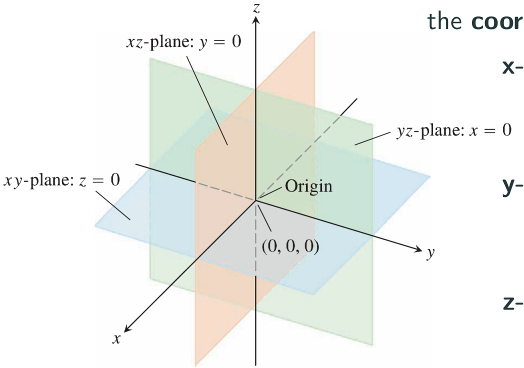

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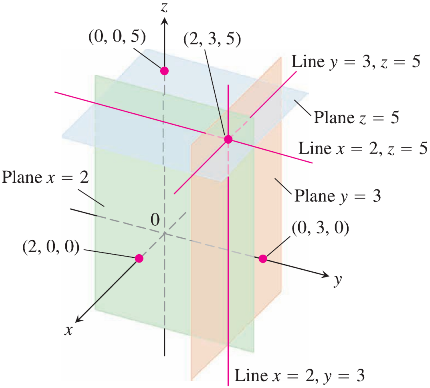

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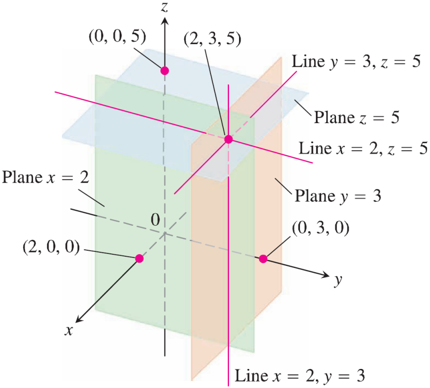

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The plane $x = 2$ is the plane perpendicular to the x-axis at $x = 2$. The plane $y = 3$ is the plane perpendicular to the y-axis at $y = 3$.

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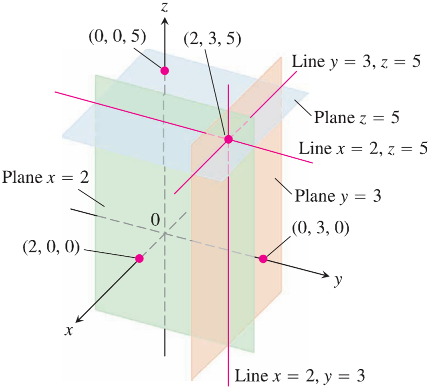

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The planes $x = 2$ and $y = 3$ intersect in a line parallel to the z-axis. This line is described by the pair of equations $x = 2, y = 3$. A point (x, y, z) lies on the line if and only if $x = 2$ and $y = 3$.

Analytic Geometry in Three Dimensions

EXAMPLE:
Interpret the following equations and inequalities geometrically:

(a) $z \geq 0$

(b) $x = -3$

(c) $z = 0, x \leq 0, y \geq 0$

(d) $x \geq 0, y \geq 0, z \geq 0$

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(f) The line in which the planes $y = -2$ and $z = 2$ intersect (parallel to x-axis).

Analytic Geometry in Three Dimensions

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What points $P(x, y, z)$ satisfy the equations:

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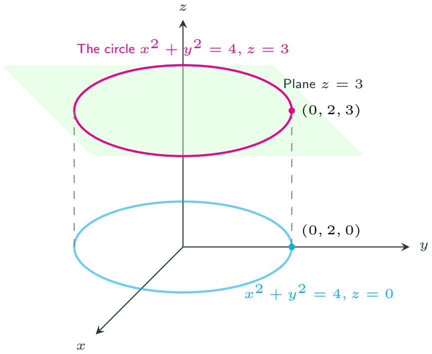
We call this set of points:

- "The circle $x^2 + y^2 = 4$ in the plane $z = 3$".
- Or simply, "the circle $x^2 + y^2 = 4, z = 3$".



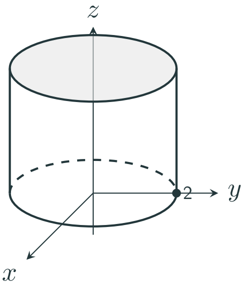
Surfaces in Three Dimensions: Cylinders

EXAMPLE: Identifying Cylindrical Surfaces

$$x^2 + y^2 = 4$$

Represents all points on the vertical **circular cylinder** containing the circle $x^2 + y^2 = 4$ in the xy-plane.

- **Radius:** 2
- **Axis:** z-axis



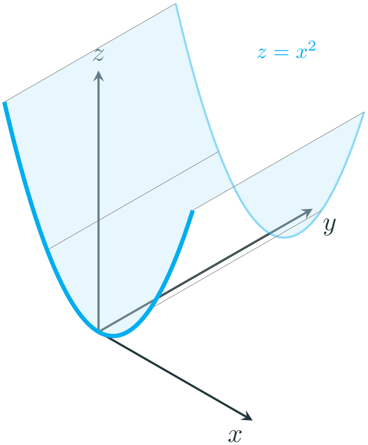
Analytic Geometry in Three Dimensions

EXAMPLE: The Parabolic Cylinder $z = x^2$

In 3D space, since the variable y is missing from the equation $z = x^2$, the surface extends infinitely along the y-axis.

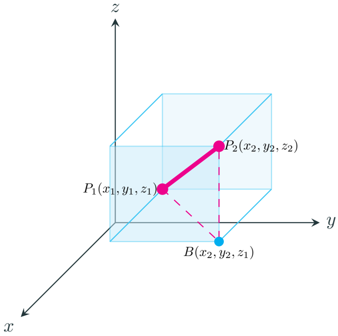
Geometric Properties:

- The cross-section in any plane $y = c$ is the parabola $z = x^2$.
- The surface is tangent to the xy-plane along the entire y-axis.
- It is a "cylinder" because it is generated by a line moving along a curve (the generating curve).



Distance Between Two Points in 3D Space

To find the **distance** between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, we consider the rectangular box with these points as opposite vertices.



The distance $|P_1P_2|$ is the diagonal of the rectangular box.

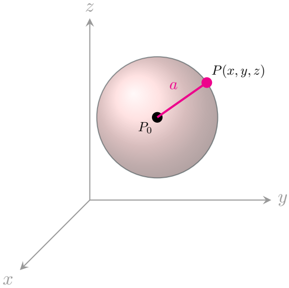
Distance Formula in 3D

The Distance Between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is: $|P_1P_2| =$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The Standard Equation for the Sphere

A sphere is the set of all points $P(x, y, z)$ in space that are at a fixed distance a from a fixed point $P_0(x_0, y_0, z_0)$.



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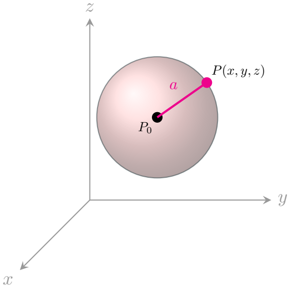

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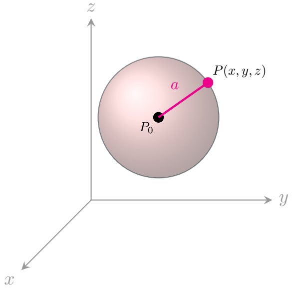

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Special Case: Origin

If the center is at the origin $(0, 0, 0)$, the equation becomes:

$$x^2 + y^2 + z^2 = a^2$$

Finding Center and Radius of a Sphere

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$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z-2)^2 = -1 + \frac{9}{4} + 4 = \frac{21}{4}$$

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Result
From this standard form, we identify:

- **Center** (x_0, y_0, z_0)**:** $(-3/2, 0, 2)$
- **Radius** a**:** $\sqrt{21}/2$

Analytic Geometry in n-Space

Euclidean n-Space

We can generalize the concept of coordinates and distance to higher dimensions.

Definition of $\mathbb{R}^n$

The set of all n-tuples of real numbers is denoted by $\mathbb{R}^n$:

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}\}$$

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Distance in n-Space

The distance between two points $P(x_1, \ldots, x_n)$ and $Q(y_1, \ldots, y_n)$ in $\mathbb{R}^n$ is defined as:

$$d(P, Q) = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}$$

Analytic Geometry in n-Space

Describing Sets in the Plane, 3-Space, and n-Space

Neighbourhood of a Point

A **neighbourhood** of a point P in $\mathbb{R}^n$ is a set of the form:
$B_r(P) = \{Q \in \mathbb{R}^n :$ distance from Q to $P < r\}$ for some radius $r > 0$.

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- **For $n = 1$:** If $p \in \mathbb{R}$, then $B_r(p)$ is the **open interval** $(p - r, p + r)$ centred at p.

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- **For** $n = 3$**:** $B_r(P)$ is the **open ball** of radius r centred at point P.

Describing Sets: Open Sets

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Universal Open Sets

The whole space $\mathbb{R}^n$ is an open set. For technical reasons, the **empty set** $\emptyset$ is also considered to be open.

Describing Sets: Closed Sets

The Complement of a Set

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Describing Sets: Closed Sets

The Complement of a Set

The **complement**, S^c, of a set S in $\mathbb{R}^n$ is the set of all points in $\mathbb{R}^n$ that do not belong to S.

Example: In $\mathbb{R}^2$, the complement of $\{(x, y) : x > 0\}$ is $\{(x, y) : x \leq 0\}$.

Definition: Closed Set

A set is said to be **closed** if its complement is open.

- Sets defined by **nonstrict inequalities** (using $\geq$ and $\leq$) are typically closed.
- Closed intervals are closed sets in $\mathbb{R}$.

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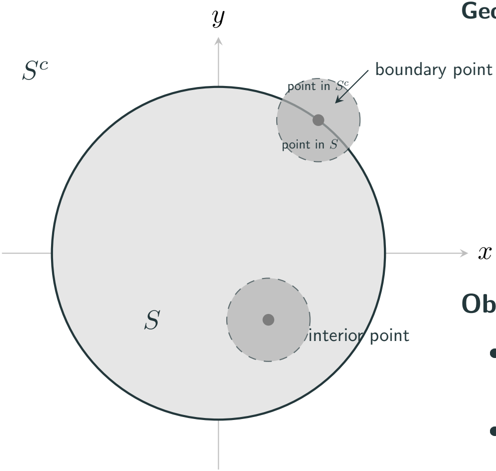
The "Clopen" Property

The whole space $\mathbb{R}^n$ and the empty set $\emptyset$ are both open and closed in $\mathbb{R}^n$. They are the **only** sets with this property.

Interior, Exterior, and Boundary

Example: The Closed Disk $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$



Geometric Components:

Boundary: bdry(S) is the circle $x^2 + y^2 = 1$.

Interior: int(S) is the **open disk** $x^2 + y^2 < 1$.

Exterior: ext(S) is the **open set** $x^2 + y^2 > 1$.

Observations

- The shaded neighbourhood of the **interior point** is entirely within S.
- The neighbourhood of the **boundary point** contains points from both S and S^c.

Introduction to Vectors

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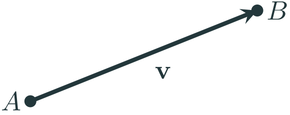

The vector $\mathbf{v} = \vec{AB}$

Magnitude:

The magnitude of the vector $\mathbf{v}$ is the length of the arrow and is denoted by:

$$|\mathbf{v}| \quad \text{or} \quad |\vec{AB}|$$

Properties of Vectors: Equality

When are two vectors considered the same?

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Equivalence Principle
The arrows used to draw vectors represent the same vector if they:

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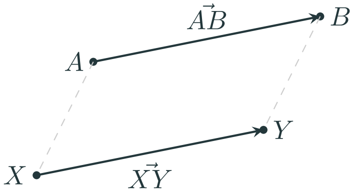
This is true **regardless of the initial point**.



$$\vec{AB} = \vec{XY}$$

Note: These two arrows have different initial points but represent the same geometric vector.

Vector Components in the Plane

Component Form of a Vector

A vector $\overrightarrow{AP}$ with initial point $A(a, b)$ and terminal point $P(p, q)$ has specific horizontal and vertical displacements.

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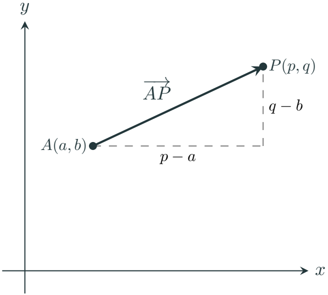

Magnitude and Slope

- $|\overrightarrow{AP}| = \sqrt{(p-a)^2 + (q-b)^2}$
- slope of $\overrightarrow{AP} = \frac{q-b}{p-a}$

- **x-component:** $p - a$
- **y-component:** $q - b$

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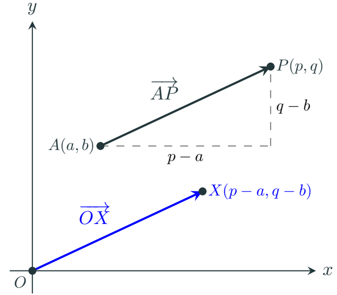

$\overrightarrow{AP} = \overrightarrow{OX}$ *if and only if they have the same x and y components.*

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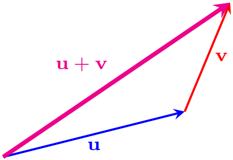
Vector Addition

The following two methods yield the same vector, denoted $\mathbf{u} + \mathbf{v}$, and called **the sum of $\mathbf{u}$ and $\mathbf{v}$**.

1. Triangle Law

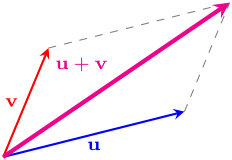
(Head-to-Tail)



$\mathbf{v}$ is moved to the end of $\mathbf{u}$

2. Parallelogram Law

(Diagonal of the Parallelogram)



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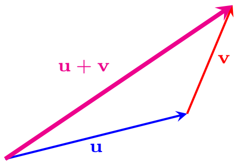

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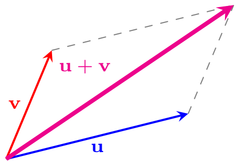

Diagonal of the parallelogram

Algebraic Consistency

In both cases, if $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, the resulting vector is:

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1,\ u_2 + v_2 \rangle$$

Properties of Vectors: Scalar Multiplication

Scalar Multiple of a Vector

If $\mathbf{v}$ is a vector and t is a real number (scalar), the **scalar multiple $t\mathbf{v}$** is a vector characterized by:

- **Magnitude:** $|t|$ times the magnitude of $\mathbf{v}$.
- **Direction:**
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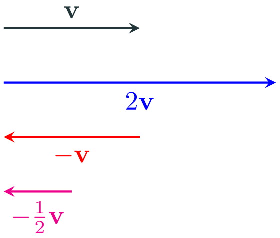

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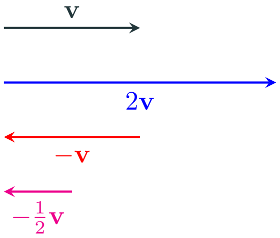


The Zero Vector

If $t = 0$, then $t\mathbf{v}$ has zero length and no particular direction. It is called

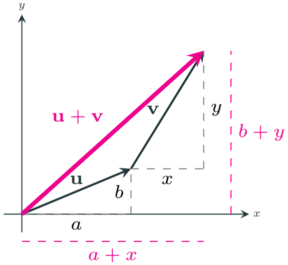
the **zero vector**, denoted by $\mathbf{0}$.

Vector Operations in Components

Suppose that $\mathbf{u}$ has components a and b, and that $\mathbf{v}$ has components x and y.

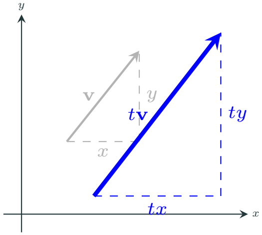
Vector Addition

$\mathbf{u} + \mathbf{v}$ has components $a + x$ and $b + y$.



Scalar Multiplication

$t\mathbf{v}$ has components tx and ty.



Algebraic Summary

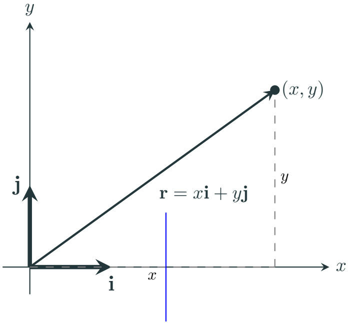
If $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle x, y \rangle$, then:

- $\mathbf{u} + \mathbf{v} = \langle a + x,\ b + y \rangle$
- $t\mathbf{v} = \langle tx,\ ty \rangle$

Standard Basis Vectors in the Plane

In $\mathbb{R}^2$, we single out two particular vectors for special attention:

- **Vector i:** From the origin to $(1, 0)$.
- **Vector j:** From the origin to $(0, 1)$.



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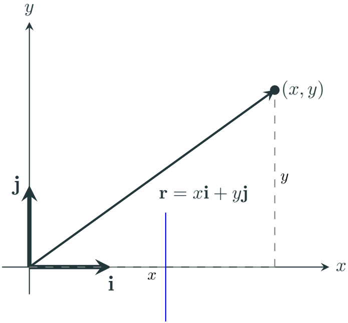
These are the **standard basis vectors**. Any vector $\mathbf{r} = \langle x, y \rangle$ can be expressed as a linear combination:

$$\mathbf{r} = \langle x, y \rangle = x\mathbf{i} + y\mathbf{j}$$



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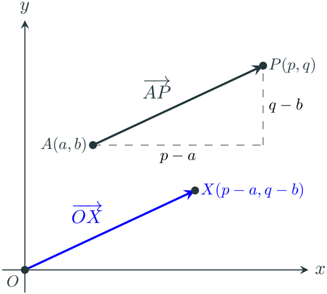
Vectors: General Linear Combinations

More generally, any vector $\overrightarrow{AP}$ from an initial point $A(a,b)$ to a terminal point $P(p,q)$ can be expressed in two equivalent forms:



1. **Component List:**

$$\overrightarrow{AP} = \langle p-a,\ q-b\rangle$$

2. **Linear Combination:**

$$\overrightarrow{AP} = (p-a)\mathbf{i} + (q-b)\mathbf{j}$$

Algebra of Vectors and Unit Vectors

Sums and scalar multiples of vectors are easily expressed in terms of the standard basis vectors $\mathbf{i}$ and $\mathbf{j}$.

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Normalizing a Vector
For any nonzero vector $\mathbf{v}$, we can form a **unit vector** $\hat{\mathbf{v}}$ in the same direction by multiplying $\mathbf{v}$ by the reciprocal of its length:

$$\hat{\mathbf{v}} = \left(\frac{1}{|\mathbf{v}|}\right)\mathbf{v}$$

Example: Vectors as Linear Combinations

EXAMPLE:

Given $A = (2, -1)$, $B = (-1, 3)$, and $C = (0, 1)$, express the following as linear combinations of $\mathbf{i}$ and $\mathbf{j}$:

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(b) $\vec{BC}$

(c) $\vec{AC}$

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(f) Since $|\vec{AB}| = \sqrt{(-3)^2 + 4^2} = 5$, the unit vector is:

$$\hat{\mathbf{u}} = \frac{\vec{AB}}{|\vec{AB}|} = -\frac{\mathbf{3}}{\mathbf{5}}\mathbf{i} + \frac{\mathbf{4}}{\mathbf{5}}\mathbf{j}$$

Properties of Vector Operations

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and a, b be scalars. The following properties hold for vector addition and scalar multiplication:

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Scalar Multiplication

Properties of Vector Operations

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and a, b be scalars. The following properties hold for vector addition and scalar multiplication:

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1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
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9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Vectors in Three-Dimensional Space

The concepts developed for $\mathbb{R}^2$ extend naturally to $\mathbb{R}^3$ by introducing a third dimension.

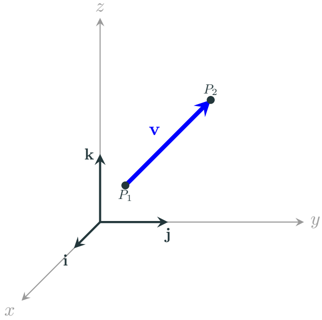
Standard Basis in $\mathbb{R}^3$

We define three unit vectors along the axes:

- $\mathbf{i} = \langle 1, 0, 0 \rangle$
- $\mathbf{j} = \langle 0, 1, 0 \rangle$
- $\mathbf{k} = \langle 0, 0, 1 \rangle$

Any vector $\mathbf{v}$ from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is given by:

$$\begin{aligned}\mathbf{v} &= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \\ &= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}\end{aligned}$$



Magnitude in $\mathbb{R}^3$

$$|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example: Vector Operations in $\mathbb{R}^3$

EXAMPLE:
Given the vectors $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, find the following:

- $\mathbf{u} + \mathbf{v}$
- $\mathbf{u} - \mathbf{v}$
- $3\mathbf{u} - 2\mathbf{v}$
- $|\mathbf{u}|$ and $|\mathbf{v}|$
- Unit vector $\hat{\mathbf{u}}$ in the direction of $\mathbf{u}$

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Solution:

- $\mathbf{u} + \mathbf{v} = (2+3)\mathbf{i} + (1-2)\mathbf{j} + (-2-1)\mathbf{k} = \mathbf{5i} - \mathbf{j} - \mathbf{3k}$

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Solution:

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- **Magnitudes:**
  - $|\mathbf{u}| = \sqrt{4+1+4} = \mathbf{3}$
  - $|\mathbf{v}| = \sqrt{9+4+1} = \sqrt{\mathbf{14}}$

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- **Magnitudes:**
    - $|\mathbf{u}| = \sqrt{4+1+4} = \mathbf{3}$
    - $|\mathbf{v}| = \sqrt{9+4+1} = \sqrt{\mathbf{14}}$
- **Unit Vector:**

$$\hat{\mathbf{u}} = \left(\frac{1}{|\mathbf{u}|}\right)\mathbf{u} = \frac{\mathbf{2}}{\mathbf{3}}\mathbf{i} + \frac{\mathbf{1}}{\mathbf{3}}\mathbf{j} - \frac{\mathbf{2}}{\mathbf{3}}\mathbf{k}$$