# MAT124 MATHEMATICS II

Analytic Geometry in 3-Dimensional Space and Vectors

# Outline

# The Dot Product

# The Dot Product and Projections

### Dot Product in $\mathbb{R}^2$

Given two vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ in $\mathbb{R}^2$, their **dot product** $\mathbf{u} \cdot \mathbf{v}$ is the sum of the products of their corresponding components:

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**Generalization: Dot Product in $\mathbb{R}^3$**

Similarly, for vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ in $\mathbb{R}^3$:

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**Algebraic Properties**

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and $t$ be a real scalar:

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- $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$ *(relationship to magnitude)*

# Geometric Interpretation of the Dot Product

**Theorem (Geometric Interpretation)**
*If* $\theta$ *is the angle between the directions of* $\mathbf{u}$ *and* $\mathbf{v}$ *(*$0 \leq \theta \leq \pi$*), then:*

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$$

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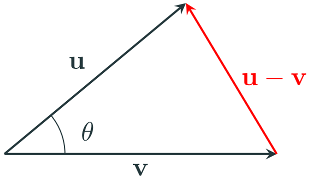

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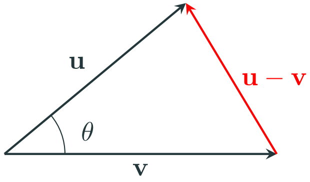


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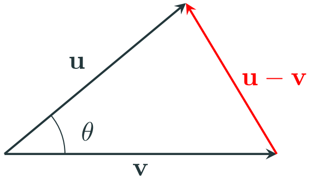


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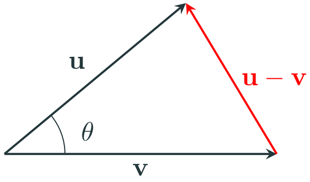


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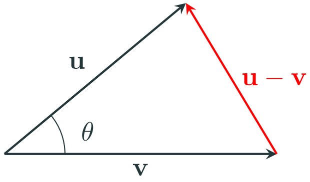


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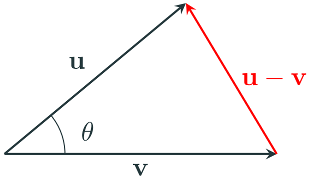


Comparing (1) and (2) yields: $|\mathbf{u}||\mathbf{v}| \cos \theta = \mathbf{u} \cdot \mathbf{v}$.

# Calculating the Angle Between Vectors

**EXAMPLE:**
Find the angle $\theta$ between the vectors:

$$\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \quad \text{and} \quad \mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

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$$\begin{aligned}
\theta &= \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) \\
&= \cos^{-1}\left(\frac{(2)(3) + (1)(-2) + (-2)(-1)}{3\sqrt{14}}\right) \\
&= \cos^{-1}\left(\frac{6 - 2 + 2}{3\sqrt{14}}\right) = \cos^{-1}\left(\frac{6}{3\sqrt{14}}\right) \\
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**Result**
The angle between the two vectors is approximately $57.69^\circ$.

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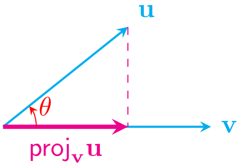
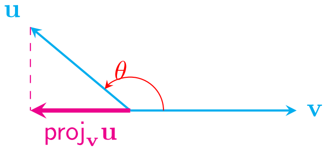
The vector projection $\text{proj}_{\mathbf{v}}\mathbf{u}$ adapts its direction based on the angle $\theta$, while the formula remains constant.




Length $= |\mathbf{u}| \cos\theta$

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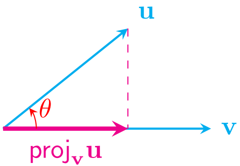


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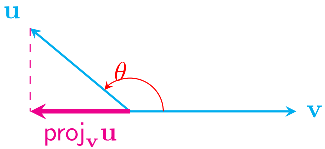


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$$\text{proj}_{\mathbf{v}}\mathbf{u} = (|\mathbf{u}|\cos\theta)\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

$$|\mathbf{u}|\cos\theta = \frac{|\mathbf{u}||\mathbf{v}|\cos\theta}{|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$$

# Projection and Scalar Component

The relationship between the vector projection and the scalar component is fundamental for decomposing vectors.

**Vector Projection**

The **vector projection** of $\mathbf{u}$ onto $\mathbf{v}$ is the vector:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

**Scalar Component**

The **scalar component** of $\mathbf{u}$ in the direction of $\mathbf{v}$ is the scalar:

$$|\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}$$

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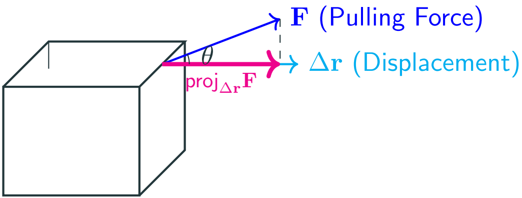


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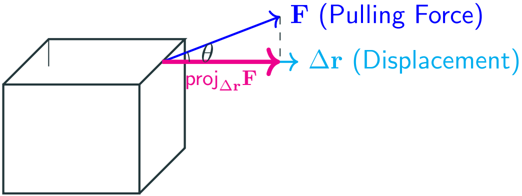


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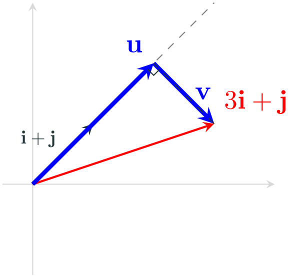
**Physical Work** Work $(W)$ is defined as the dot product of force and displacement:

$$W = \mathbf{F} \cdot \Delta\mathbf{r} = (|\mathbf{F}| \cos\theta)|\Delta\mathbf{r}|$$

where $|\mathbf{F}| \cos\theta$ is the **scalar component** of force.

# Example: Decomposing a Vector into Orthogonal Components

**EXAMPLE:**
Express the vector $3\mathbf{i} + \mathbf{j}$ as a sum of vectors $\mathbf{u} + \mathbf{v}$, where $\mathbf{u}$ is parallel to $\mathbf{i} + \mathbf{j}$ and $\mathbf{v}$ is perpendicular to $\mathbf{u}$.



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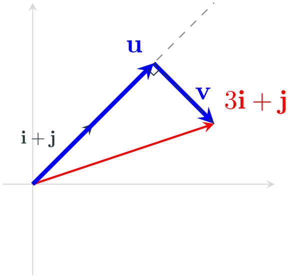
$$= \text{proj}_{(\mathbf{i}+\mathbf{j})}(3\mathbf{i}+\mathbf{j})$$



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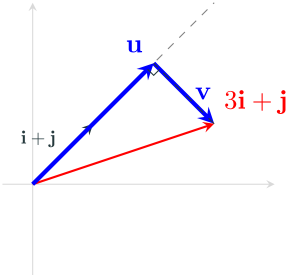

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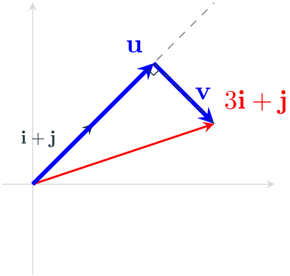

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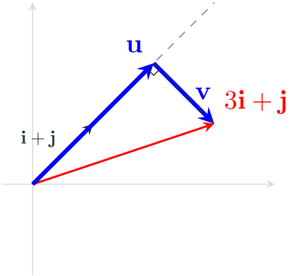

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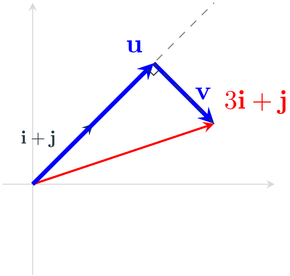

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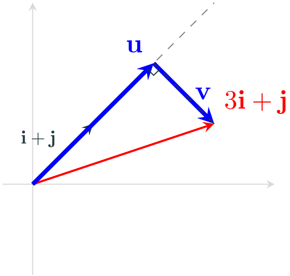
**Method I:** Note that $\mathbf{u}$ must be the vector projection of $3\mathbf{i}+\mathbf{j}$ in the direction of $\mathbf{i}+\mathbf{j}$:

$$\begin{aligned}\mathbf{u} &= \text{proj}_{(\mathbf{i}+\mathbf{j})}(3\mathbf{i}+\mathbf{j}) \\ &= \frac{(3\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{j})}{|\mathbf{i}+\mathbf{j}|^2}(\mathbf{i}+\mathbf{j}) \\ &= \frac{3(1)+1(1)}{(\sqrt{1^2+1^2})^2}(\mathbf{i}+\mathbf{j}) = \frac{4}{2}(\mathbf{i}+\mathbf{j}) \\ &= \mathbf{2i}+\mathbf{2j}\end{aligned}$$




Then,

$$\mathbf{v} = (3\mathbf{i}+\mathbf{j}) - \mathbf{u} = (3\mathbf{i}+\mathbf{j}) - (2\mathbf{i}+2\mathbf{j}) = \mathbf{i}-\mathbf{j}$$

# Example: Decomposing a Vector into Orthogonal Components

**EXAMPLE:**
Express the vector $3\mathbf{i} + \mathbf{j}$ as a sum of vectors $\mathbf{u} + \mathbf{v}$, where $\mathbf{u}$ is parallel to $\mathbf{i} + \mathbf{j}$ and $\mathbf{v}$ is perpendicular to $\mathbf{u}$.

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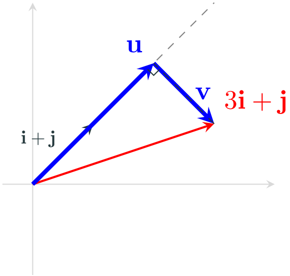


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**Verification**
$\mathbf{u} \cdot \mathbf{v} = (2)(1) + (2)(-1) = 0$
Therefore, $\mathbf{u} \perp \mathbf{v}$.

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Express the vector $3\mathbf{i}+\mathbf{j}$ as a sum of vectors $\mathbf{u}+\mathbf{v}$, where $\mathbf{u}$ is parallel to $\mathbf{i}+\mathbf{j}$ and $\mathbf{v}$ is perpendicular to $\mathbf{u}$.

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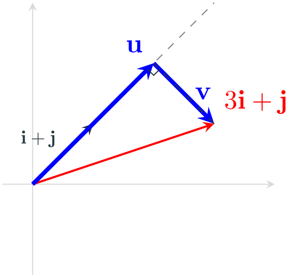
- Since $\mathbf{u}$ is parallel to $\mathbf{i}+\mathbf{j}$, we have $\mathbf{u}=t(\mathbf{i}+\mathbf{j})$.



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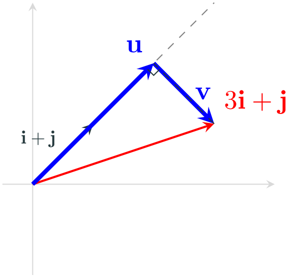

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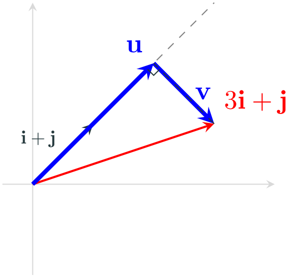


We want $\mathbf{u}+\mathbf{v}=3\mathbf{i}+\mathbf{j}$. Taking the dot product with $\mathbf{i}+\mathbf{j}$:

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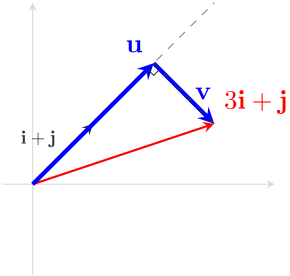


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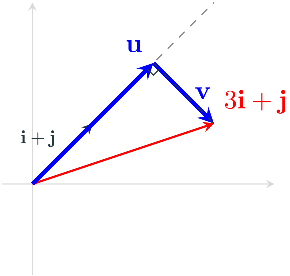


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$$\begin{aligned} \mathbf{u} \cdot (\mathbf{i} + \mathbf{j}) + \mathbf{v} \cdot (\mathbf{i} + \mathbf{j}) &= (3\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) \\ t(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) + 0 &= 3(1) + 1(1) \\ t(1^2 + 1^2) &= 4 \end{aligned}$$

# Example: Decomposing a Vector into Orthogonal Components

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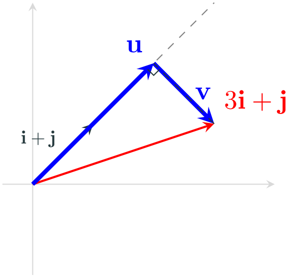


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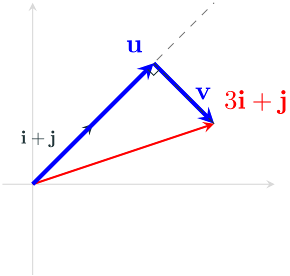
$$t(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{j})+0=3(1)+1(1)$$
$$t(1^2+1^2)=4$$
$$2t=4 \implies \mathbf{t}=\mathbf{2}$$




Therefore:

- $\mathbf{u}=2(\mathbf{i}+\mathbf{j})=\mathbf{2i}+\mathbf{2j}$
- $\mathbf{v}=(3\mathbf{i}+\mathbf{j})-\mathbf{u}=\mathbf{i}-\mathbf{j}$

# Vectors in $n$-Space

## Vectors in $n$-Space

Vectors in $\mathbb{R}^n$ are expressed as linear combinations of the $n$ unit vectors:

- $\mathbf{e}_1$ from the origin to the point $(1, 0, 0, \ldots, 0)$
- $\mathbf{e}_2$ from the origin to the point $(0, 1, 0, \ldots, 0)$

$\vdots$

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**Standard Basis in $\mathbb{R}^n$**

These vectors constitute a *standard basis* in $\mathbb{R}^n$. The $n$-vector $\mathbf{x}$ with components $x_1, x_2, \ldots, x_n$ is expressed as:

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**Magnitude in $n$-Space**

The length (magnitude) of $\mathbf{x}$ is defined as:

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

# Angle Between Vectors in $n$-Space

The geometric concept of the angle between two vectors extends naturally to higher dimensions using the dot product.

**Angle Formula in $\mathbb{R}^n$**
The angle $\theta$ between two non-zero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is given by:

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**The Dot Product in $n$-Space**

Where the dot product $\mathbf{x}\cdot\mathbf{y}$ is defined as the sum of the products of their corresponding components:

$$\mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

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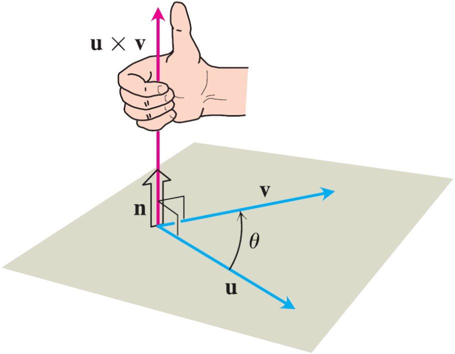
For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is the unique vector satisfying three specific conditions:

(i) **Orthogonality:** $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$ and $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$.

(ii) **Magnitude:** $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$, where $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$ $(0 \leq \theta \leq \pi)$.

(iii) **Orientation:** $\mathbf{u}, \mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a **right-handed triad**.




*The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.*

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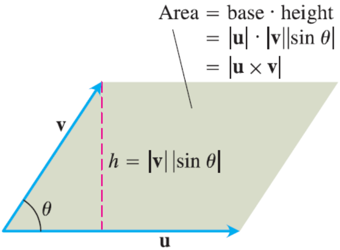

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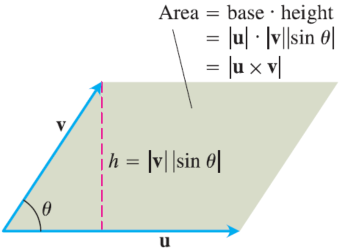


## Parallel Vectors

Two nonzero vectors are parallel if and only if their cross product is the zero vector.

# Properties of the Cross Product

The cross product satisfies several algebraic properties, most notably its non-commutative nature:

## Algebraic Properties

(1) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
*(Anti-commutative)*

(2) $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$

(3) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

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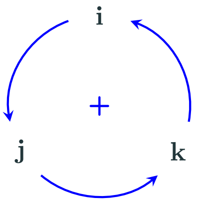
## Standard Basis Relations

- $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$
- $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$
- $\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$




Cyclic Permutation

**Geometric Insight**
The cross product of any two consecutive unit vectors in the circle yields the third one.

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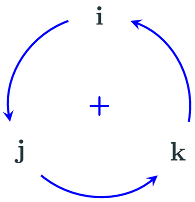


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**Geometric Insight**

The cross product of any two consecutive unit vectors in the circle yields the third one.

$$(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \times (-2\mathbf{j} + 5\mathbf{k}) = ((1)(5) - (-2)(-3))\mathbf{i} + ((-3)(0) - (2)(5))\mathbf{j} + ((2)(-2) - (1)(0))\mathbf{k}$$

# Components of the Cross Product

The cross product of two vectors in $\mathbb{R}^3$ can be calculated directly from their components.

**Algebraic Definition**
If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then:

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

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**Calculation Aid (Determinant Form):** The formula above is equivalent to the symbolic determinant expansion along the first row:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

# Determinants and Expansion in Minors

The calculation of a $3 \times 3$ determinant is performed by expanding it into smaller $2 \times 2$ determinants called **minors**.

**Expansion along the First Row**

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

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$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

The pattern of $+$ and $-$ signs for a $3 \times 3$ determinant.

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**Alternative Expansion (along $b, e, h$):**

$$-b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + e \begin{vmatrix} a & c \\ g & i \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

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Evaluate the following determinant by expanding along the second column:

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$$= \mathbf{-23}$$

# Properties of Determinants

**Understanding how row operations affect a determinant is crucial for simplifying calculations.**

(i) **Row Interchange:** If two rows of a determinant are interchanged, the determinant changes sign:

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$$\begin{vmatrix} a & b & c \\ d + ta & e + tb & f + tc \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

# The Cross Product as a Determinant

The cross product of two vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ can be efficiently computed using a symbolic determinant.

## Algebraic Expansion

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

## Symbolic Determinant Form

The components above are the results of expanding the following $3 \times 3$ determinant along its first row:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

*Note the alternating sign $(-)$ for the $\mathbf{j}$ component in the expansion.*

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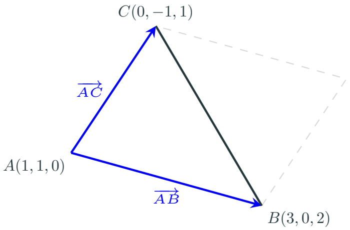

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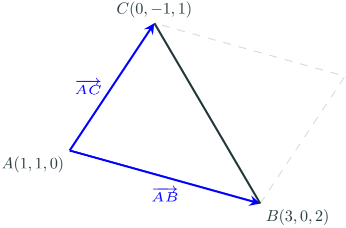
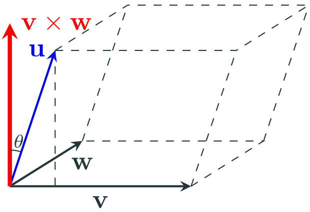



The area is half the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\text{Area} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}|2\mathbf{i} - \mathbf{j} + 2\mathbf{k} \times (-\mathbf{i} - 2\mathbf{j} + \mathbf{k})|$$
$$= \frac{1}{2}|3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}| = \frac{1}{2}\sqrt{9+16+25}$$
$$= \frac{\mathbf{5}}{\mathbf{2}}\sqrt{\mathbf{2}} \text{ square units}$$

# The Scalar Triple Product and Volume

A **parallelepiped** is the three-dimensional analogue of a parallelogram, spanned by three vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ meeting at a vertex.

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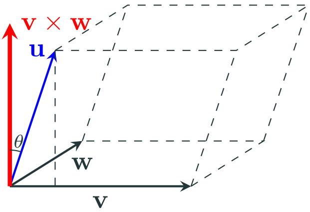


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**Volume Calculation**

The volume of the parallelepiped is:

$$\begin{aligned}\text{Volume} &= |\mathbf{u}||\mathbf{v} \times \mathbf{w}||\cos\theta| \\ &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \text{ cubic units}\end{aligned}$$

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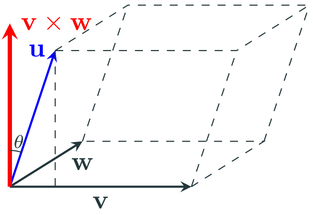


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**Scalar Triple Product**

The quantity $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is called the **scalar triple product** of vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$.

# The Scalar Triple Product as a Determinant

The scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is easily expressed in terms of a $3 \times 3$ determinant.

**Determinant Representation**
If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, and similar representations hold for $\mathbf{v}$ and $\mathbf{w}$, then:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

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**Cyclic Property**

Using the properties of determinants, it is easily verified that:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$

# Coplanar Vectors in 3-Space

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Vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are coplanar if and only if the parallelepiped they span has zero volume:

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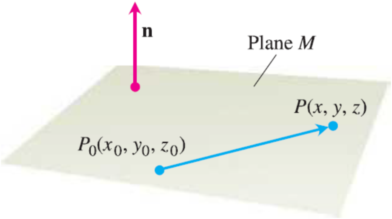
Three vectors are certainly coplanar if any one of them is the **zero vector** $\mathbf{0}$ or any pair of them is **parallel**. If neither of these degenerate cases holds, then the coplanarity condition implies that one vector can be expressed as a **linear combination** of the other two.

# Planes in 3-Space

# Planes in 3-Space: The Point-Normal Equation

A plane in $\mathbb{R}^3$ is uniquely determined by a point $P_0$ on the plane and a vector $\mathbf{n}$ perpendicular to it.




Let $P_0(x_0, y_0, z_0)$ be a fixed point with position vector $\mathbf{r}_0$. If $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is a **normal vector**, then for any point $P(x, y, z)$ on the plane, $\overrightarrow{P_0P} \perp \mathbf{n}$.

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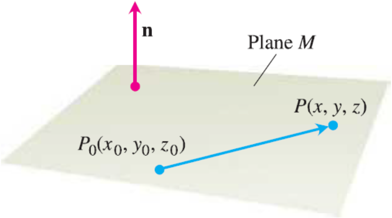


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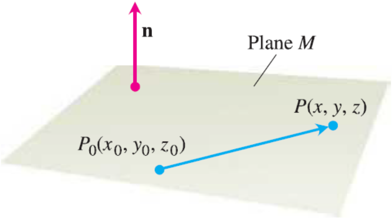


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$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

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$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

*The normal vector $\mathbf{n}$ is orthogonal to every vector lying in the plane.*

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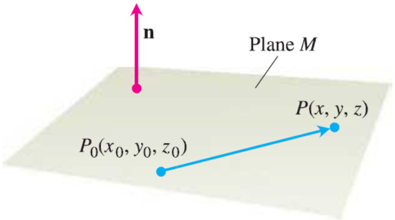


**Standard Equation**

$Ax + By + Cz = D$, where

$$D = Ax_0 + By_0 + Cz_0$$

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# Example: Recognizing and Writing Equations of Planes

Analyzing different forms of plane equations based on their geometric properties:

(a) **Origin and Normal:** The equation $2x - 3y - 4z = 0$ represents a plane passing through the **origin** with normal vector:

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(d) **General Form:** The plane with equation $2x + y + 3z = 6$ has a normal vector $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.
To find where the plane intersects the axes, we set two variables to zero:

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- Since the equation is **independent of** $z$, the plane is **parallel to the** $z$**-axis**.
- In 3-space, this is a vertical plane containing the line $2x - y = 1$ of the $xy$-plane.

(d) **General Form:** The plane with equation $2x + y + 3z = 6$ has a normal vector $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.
To find where the plane intersects the axes, we set two variables to zero:
- **x-intercept:** Set $y = z = 0 \implies 2x = 6 \implies x = 3$.
  Point: **(3, 0, 0)**.

## Example: Recognizing and Writing Equations of Planes

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  Point: **(3, 0, 0)**.
- **y-intercept:** Set $x = z = 0 \implies y = 6$. Point: **(0, 6, 0)**.
- **z-intercept:** Set $x = y = 0 \implies 3z = 6 \implies z = 2$.
  Point: **(0, 0, 2)**.

# The Intercept Form of the Plane Equation

If a plane intersects the coordinate axes at non-zero points, its equation can be expressed in a highly symmetric and useful form.

In general, if $a, b$, and $c$ are all nonzero, the plane with intercepts $a, b$, and $c$ on the coordinate axes has the equation:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

This is known as the **intercept form** of the equation of the plane.

*The intercepts correspond to the points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$ on the $x$, $y$, and $z$ axes respectively.*