MAT124 MATHEMATICS II

Lines, Planes, and Quadric Surfaces

Outline

Planes and Lines in 3-Space

EXAMPLE:
Find an equation of the plane that passes through the three points $P(1,1,0)$, $Q(0,2,1)$, and $R(3,2,-1)$.

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Normal Vector calculation:

$$\mathbf{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$

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The Final Equation
Using point $P(1,1,0)$ and $\mathbf{n} = (-2,1,-3)$:

$$-2(x-1) + 1(y-1) - 3(z-0) = 0 \implies \mathbf{2x - y + 3z = 1}$$

Example: Intersection of Two Planes

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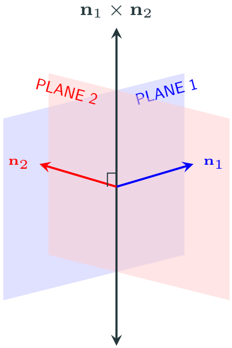

Key Insight

The line of intersection is perpendicular to both normal vectors.

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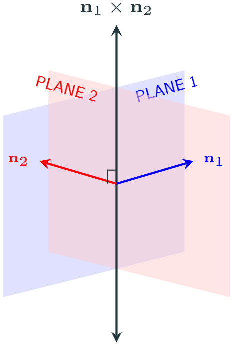
Since $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel, the planes intersect in a straight line. This line is perpendicular to both normals, so its direction vector $\mathbf{v}$ is:

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\mathbf{v} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$



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Pencil of Planes

A family of planes intersecting along a common straight line is called a **pencil of planes**.

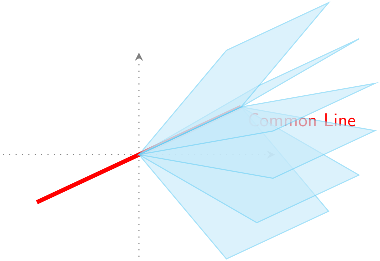
Determining a Pencil

A pencil is **uniquely** determined by any two **nonparallel** planes within it, as they share a unique line of intersection.

Given two nonparallel planes with equations:

- $A_1x + B_1y + C_1z = D_1$
- $A_2x + B_2y + C_2z = D_2$

Then, for any real number λ, the following equation represents a plane in the pencil:



General Equation of the Pencil

$(A_1x + B_1y + C_1z - D_1) + \lambda(A_2x + B_2y + C_2z - D_2) = 0$

Pencil of Planes

EXAMPLE:
Find an equation of the plane passing through the line of intersection of the two planes:

$$x + y - 2z = 6 \quad \text{and} \quad 2x - y + z = 2$$

and also passing through the point $(-2, 0, 1)$.

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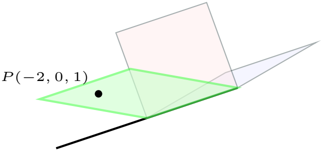
Solution: For any constant λ, the equation of the **pencil of planes** is:

$$(x + y - 2z - 6) + \lambda(2x - y + z - 2) = 0$$



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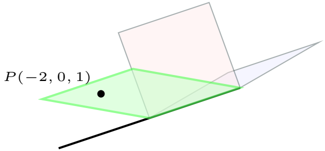
$$(x + y - 2z - 6) + \lambda(2x - y + z - 2) = 0$$

This plane passes through $(-2, 0, 1)$ if:

- $(-2 - 2 - 6) + \lambda(-4 + 1 - 2) = 0$
- $-10 + \lambda(-5) = 0 \implies \lambda = \mathbf{-2}$



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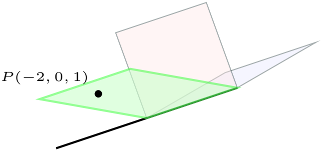
Final Simplified Equation: Substituting $\lambda = -2$ back into the pencil:

$$-3x + 3y - 4z - 2 = 0$$



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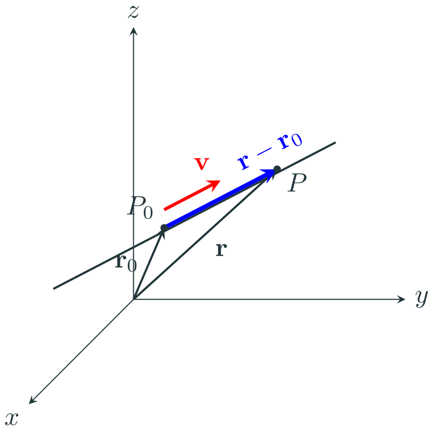
Lines in 3-Space

Lines in 3-Space: Equations and Representations

A straight line in $\mathbb{R}^3$ is determined by a point $P_0(x_0, y_0, z_0)$ and a non-zero direction vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.



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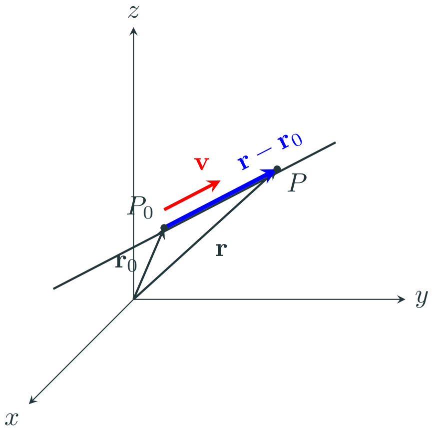

1. Vector Parametric Equation:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < \infty$$

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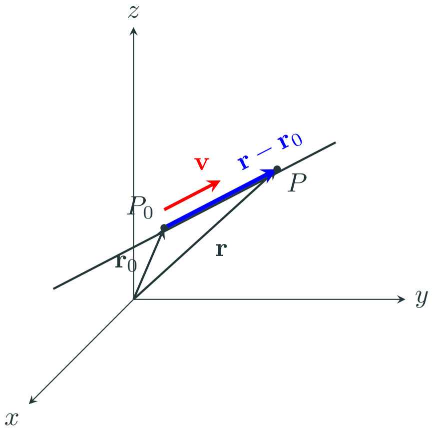

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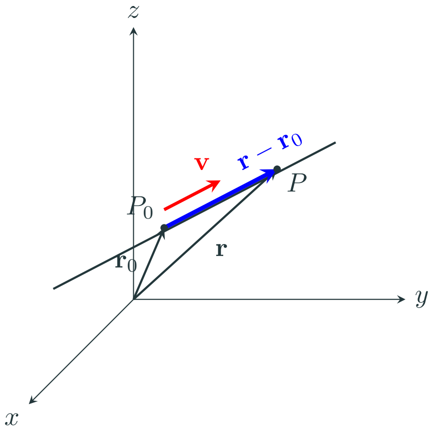

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3. Standard (Symmetric) Form: If $a, b, c \neq 0$, then:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Lines in 3-Space: Equations and Representations

The standard (symmetric) form of a line equation requires division by the components of the direction vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

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The standard form must be modified if any component of the direction vector $\mathbf{v}$ vanishes (i.e., equals zero).

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REMARK: Vanishing Components
The standard form must be modified if any component of the direction vector $\mathbf{v}$ vanishes (i.e., equals zero).

Example: Case where $c = 0$
If the z-component of the direction vector is zero, the line is parallel to the xy-plane. In this case, the equations are written as:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, \quad z = z_0$$

- This representation avoids division by zero.
- Geometrically, the condition $z = z_0$ restricts the line to a specific horizontal plane.

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By eliminating the parameter t:

$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-3}{4}$$

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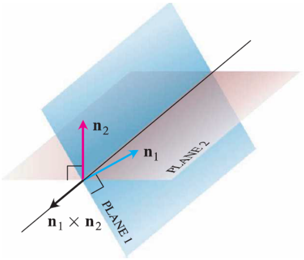

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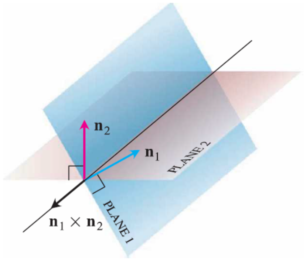

2. Finding a Point (P_0): Taking $z = 0$ and solving the system:

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Point: $(-6, 6, 0)$

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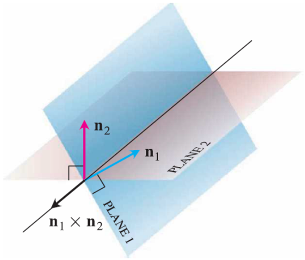

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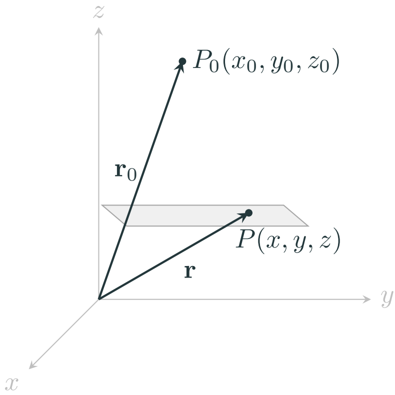
Standard Form
Using $P_0(-6, 6, 0)$ and $\mathbf{v} = (3, -2, 1)$:

$$\frac{x+6}{3} = \frac{y-6}{-2} = z$$

Distance from a Point to a Plane

Let $P_0(x_0, y_0, z_0)$ be a point and $Ax + By + Cz = D$ be a plane with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.



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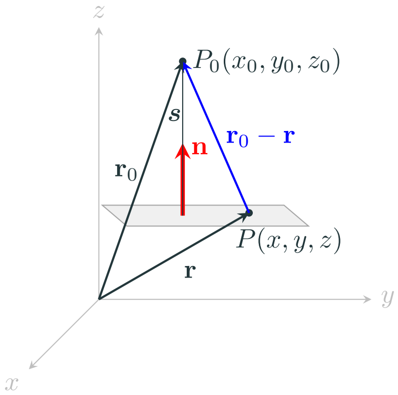

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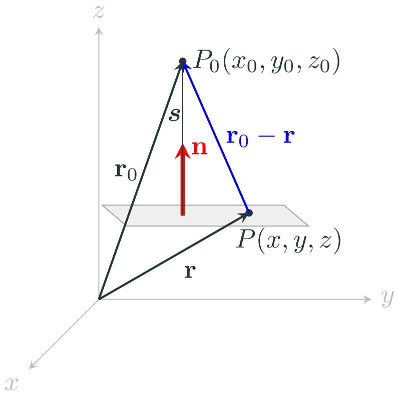

- Let P be any point on the plane and P_1 be the point closest to P_0.
- The distance s is the length of the **projection** of $\overrightarrow{PP_0}$ onto $\mathbf{n}$:

$$s = \frac{|\overrightarrow{PP_0} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|(\mathbf{r}_0 - \mathbf{r}) \cdot \mathbf{n}|}{|\mathbf{n}|}$$

Distance from a Point to a Plane

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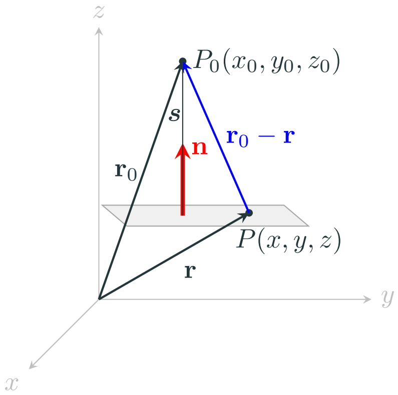

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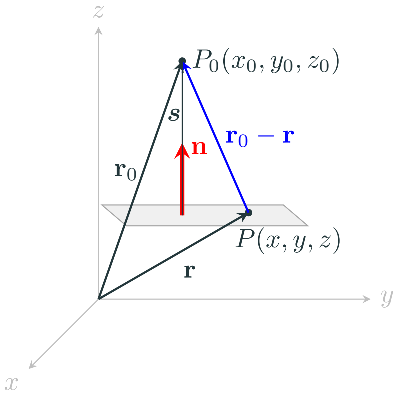
- Expanding the dot product:

$$s = \frac{|\mathbf{r}_0 \cdot \mathbf{n} - \mathbf{r} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

Distance from a Point to a Plane

Let $P_0(x_0, y_0, z_0)$ be a point and $Ax + By + Cz = D$ be a plane with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.



- Since $\mathbf{r} \cdot \mathbf{n} = D$ (plane equation), we get the final formula:

- Let P be any point on the plane and P_1 be the point closest to P_0.
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Distance Formula

$$s = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$$

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What is the distance from the point $(2, -1, 3)$ to the plane $2x - 2y - z = 9$?

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- **Identify the components:**
    - Point $P_0(x_0, y_0, z_0) = (2, -1, 3)$
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- **Apply the distance formula:**

$$s = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$$

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  - Plane coefficients: $A = 2, B = -2, C = -1, D = 9$
- **Apply the distance formula:**

$$s = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$$

- **Substitute the values:**

$$s = \frac{|2(2) - 2(-1) - 1(3) - 9|}{\sqrt{2^2 + (-2)^2 + (-1)^2}}$$

Distance from a Point to a Plane

EXAMPLE:

What is the distance from the point $(2, -1, 3)$ to the plane $2x - 2y - z = 9$?

Solution:

- **Identify the components:**
  - Point $P_0(x_0, y_0, z_0) = (2, -1, 3)$
  - Plane coefficients: $A = 2, B = -2, C = -1, D = 9$
- **Apply the distance formula:**

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- **Substitute the values:**

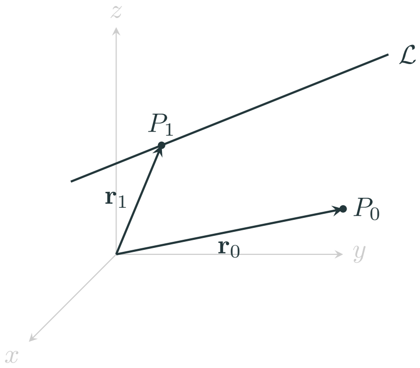
$$s = \frac{|2(2) - 2(-1) - 1(3) - 9|}{\sqrt{2^2 + (-2)^2 + (-1)^2}}$$

- **Calculate the distance:**

$$s = \frac{|4 + 2 - 3 - 9|}{\sqrt{4 + 4 + 1}} = \frac{|-6|}{3} = 2$$

Distance from a Point to a Line

The distance s from a point P_0 to a line $\mathcal{L}$ through P_1 with direction vector $\mathbf{v}$ is the length of the perpendicular segment.



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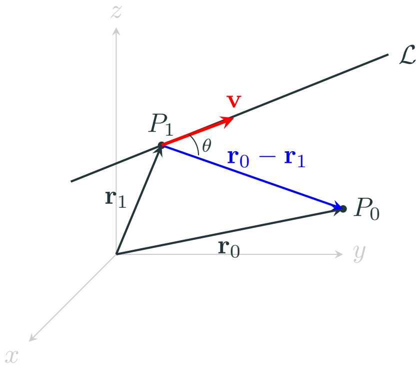

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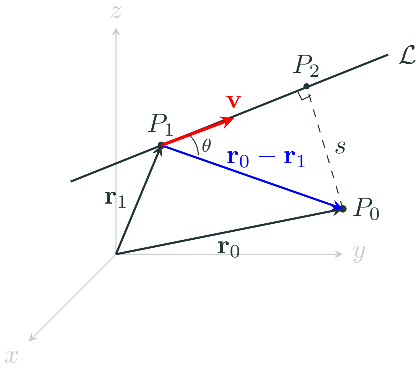

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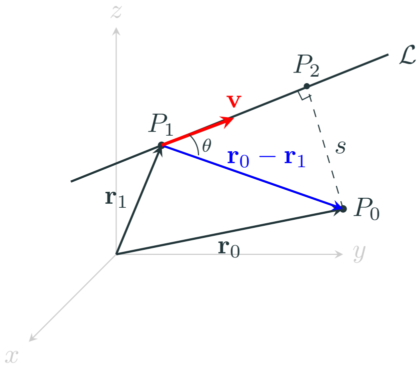

- Let P_2 be the projection of P_0 onto $\mathcal{L}$.
- From the right triangle $\triangle P_1P_2P_0$:

$$\begin{aligned} s = |P_2P_0| &= |P_1P_0| \sin\theta \\ &= |\mathbf{r}_0 - \mathbf{r}_1| \sin\theta \end{aligned}$$

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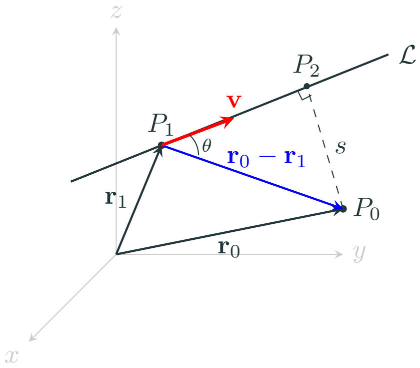

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$$s = |P_2P_0| = |P_1P_0| \sin\theta$$
$$= |\mathbf{r}_0 - \mathbf{r}_1| \sin\theta$$

- Using the cross product property:

$$|(\mathbf{r}_0 - \mathbf{r}_1) \times \mathbf{v}| = |\mathbf{r}_0 - \mathbf{r}_1||\mathbf{v}| \sin\theta$$

- Solving for s, we obtain the formula:

Distance Formula (Point to Line)

$$s = \frac{|(\mathbf{r}_0 - \mathbf{r}_1) \times \mathbf{v}|}{|\mathbf{v}|}$$

Example: Distance from a Point to a Line

EXAMPLE:
What is the distance from $P_0(2, 0, -3)$ to the line
$\mathbf{r} = \mathbf{i} + (1 + 3t)\mathbf{j} - (3 - 4t)\mathbf{k}$?

Solution:

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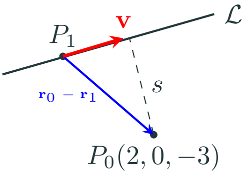

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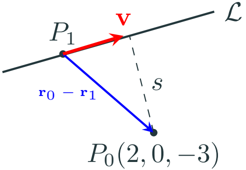

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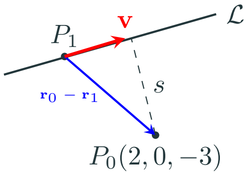

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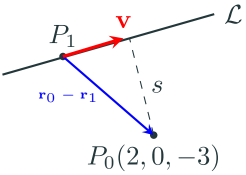

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Example: Distance from a Point to a Line

EXAMPLE:
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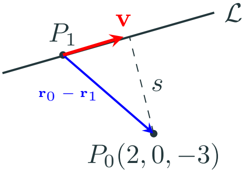

Final Result

$$s = \frac{\sqrt{41}}{5} \text{ units}$$

Quadric Surfaces

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The most general second-degree equation in three variables x, y, and z is given by:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz = J$$

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The Degenerate Case: A Pair of Planes

If the above equation can be factored in the form:

$$(A_1x + B_1y + C_1z - D_1)(A_2x + B_2y + C_2z - D_2) = 0$$

then the graph is a **pair of planes** $(A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2)$ or a single plane if the equations are identical.

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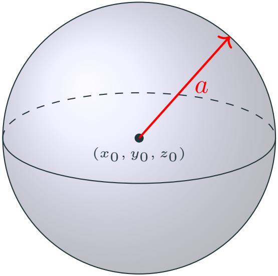
Definition: Quadric Surface
If such factorization is **not possible**, then the surface represented by the general second-degree equation is called a **quadric surface**.

Quadric Surfaces: Spheres

The simplest type of quadric surface is the sphere.

The equation of a sphere with radius a is:

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = a^2$$



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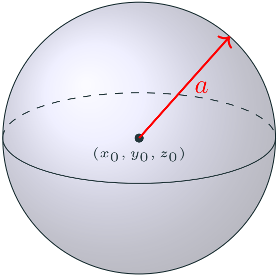


Identification Criteria

A quadratic equation in x, y, and z represents a sphere if:

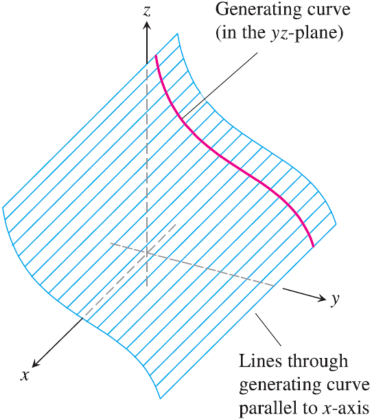
- The coefficients of x^2, y^2, and z^2 are **equal**.
- There are **no other second-degree terms** (no xy, xz, or yz).

The center can be found by **completing the squares**.

Quadric Surfaces: Cylinders

Definition
A **cylinder** is a surface generated by moving a straight line along a given planar curve while holding the line parallel to a given fixed line.

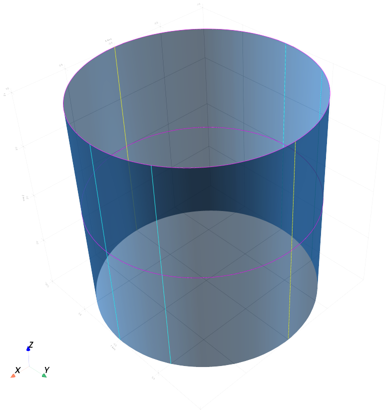
- The curve is called a **generating curve** for the cylinder.



Quadric Surfaces: Cylinders

Circular Cylinder

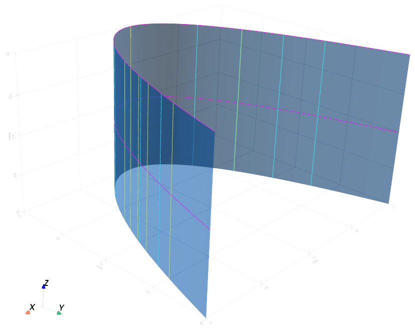
$$x^2 + y^2 = a^2$$



Generated by a circle in the xy-plane moving a line parallel to the z-axis.

Parabolic Cylinder

$$y = x^2$$



Generated by a parabola in the xz-plane moving a line parallel to the y-axis.

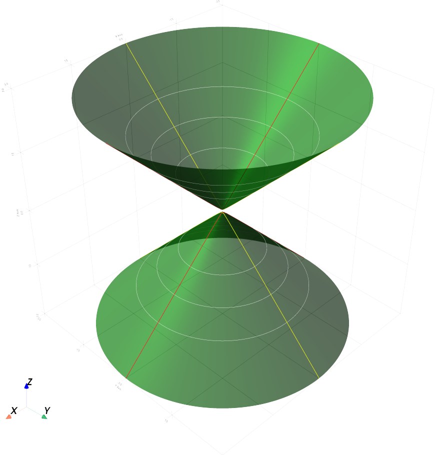
Quadric Surfaces: Elliptical Cone

An **elliptical cone** with its vertex at the origin and axis along the z-axis.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \quad \text{with } a = 1.2, b = 1.8$$

Visualization Traces:

- **Horizontal ($z = k$):** Ellipses that scale linearly with height.
- **xz-trace ($y = 0$):** Pair of intersecting lines.
- **yz-trace ($x = 0$):** Pair of intersecting lines.



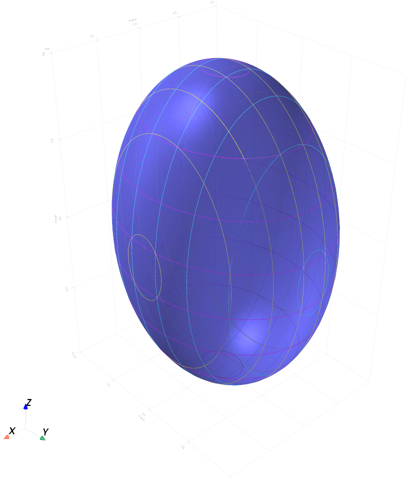
Quadric Surfaces: Ellipsoid

The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a smooth surface with elliptical traces in all coordinate planes.

- **yz-trace:** Ellipse, $x = c_1$.
- **xz-trace:** Ellipse, $y = c_2$.
- **xy-trace:** Ellipse, $z = c_3$.

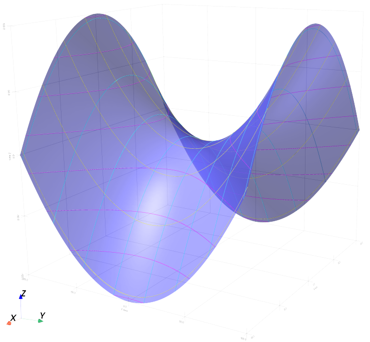


Quadric Surfaces: Hyperbolic Paraboloid (The Saddle)

The surface defined by the equation below is called a **hyperbolic paraboloid** due to its vertical parabolic traces and horizontal hyperbolic traces.

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c} \quad \text{(with } c > 0\text{)}$$

- **Traces ($x = k$):** Parabolas opening upwards ($z = \frac{y^2}{b^2} - C$).
- **Traces ($y = k$):** Parabolas opening downwards ($z = C - \frac{x^2}{a^2}$).
- **Traces ($z = k$):** Hyperbolas (for $k \neq 0$) or intersecting lines (for $k = 0$).
- **Center:** The origin $(0, 0, 0)$ is a **saddle point**.

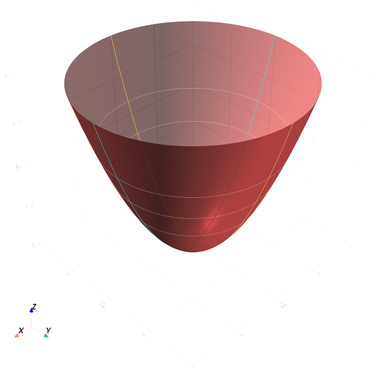


Eliptik Paraboloid

The surface defined by the equation below is called an **elliptic paraboloid**.

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

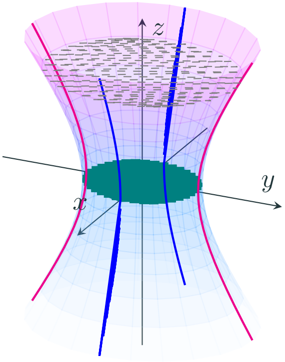
- z-traces are ellipses.
- Vertical traces in the coordinate planes are parabolas.



Quadric Surfaces: Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

- **xy-trace:** $z = 0$ (The waist of the surface).
- **xz-trace:** Hyperbola on $y = 0$ plane.
- **yz-trace:** Hyperbola on $x = 0$ plane.



Quadric Surfaces: Hyperboloid of Two Sheets

The hyperboloid of two sheets consists of two separate parts. No points exist between the vertices $(0, 0, \pm c)$.

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (a, b, c > 0)$$

Key Characteristics:

- **Vertices:** $(0, 0, c)$ and $(0, 0, -c)$.
- **xz-trace ($y = 0$):** Hyperbola $\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$.
- **yz-trace ($x = 0$):** Hyperbola $\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1$.
- **Horizontal Traces:** Ellipses for $|z| > c$; no trace for $|z| < c$.