MAT124 MATHEMATICS II

Vector Functions and Space Curves

Outline

Vector Functions of One Variable

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Such functions:

- can be used to represent **curves parametrically**;
- can be thought of as giving the **position**, at time t, of a moving point (or a particle).

Vector Functions of One Variable

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In terms of the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, the position of the particle at time t is

$$\text{position:} \quad \mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

DEFINITION Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function with domain D, and $\mathbf{L}$ a vector. We say that $\mathbf{r}$ has **limit** $\mathbf{L}$ as t approaches t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$|\mathbf{r}(t) - \mathbf{L}| < \epsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

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If $\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$, then it can be shown that $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$ precisely when

$$\lim_{t \to t_0} f(t) = L_1, \qquad \lim_{t \to t_0} g(t) = L_2, \qquad \text{and} \qquad \lim_{t \to t_0} h(t) = L_3.$$

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We omit the proof. The equation

$$\lim_{t \to t_0} \mathbf{r}(t) = \left(\lim_{t \to t_0} f(t)\right)\mathbf{i} + \left(\lim_{t \to t_0} g(t)\right)\mathbf{j} + \left(\lim_{t \to t_0} h(t)\right)\mathbf{k}$$

provides a practical way to calculate limits of vector functions.

EXAMPLE

If $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, then

$$\begin{aligned}\lim_{t\to\pi/4}\mathbf{r}(t) &= \left(\lim_{t\to\pi/4}\cos t\right)\mathbf{i} + \left(\lim_{t\to\pi/4}\sin t\right)\mathbf{j} + \left(\lim_{t\to\pi/4} t\right)\mathbf{k}\\ &= \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}.\end{aligned}$$

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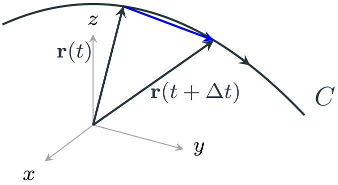
DEFINITION A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

Vector Functions of One Variable

average velocity : $\dfrac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}$



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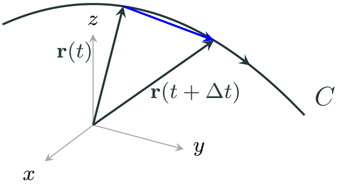

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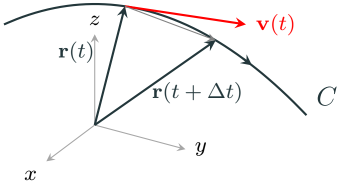

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velocity: $$\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t} = \frac{d}{dt}\mathbf{r}(t)$$

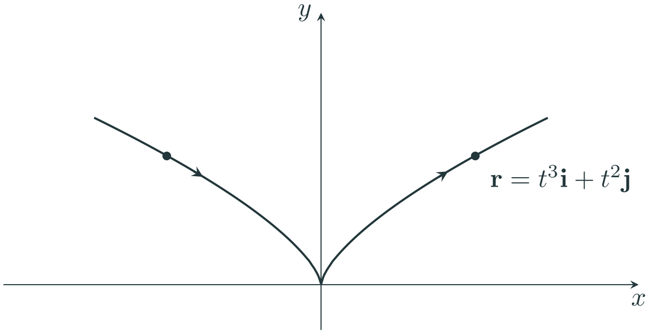
speed: $v(t) = |\mathbf{v}(t)|$

(If the limit exists, $\mathbf{r}$ is called differentiable at t.)

EXAMPLE

Consider the plane curve $\mathbf{r} = t^3\mathbf{i} + t^2\mathbf{j}$. Its component functions t^3 and t^2 have continuous derivatives of all orders. However, the curve is not **smooth** at the origin $(t = 0)$, where its velocity $\mathbf{v} = 3t^2\mathbf{i} + 2t\mathbf{j} = \mathbf{0}$.

The curve is smooth at all other points where $\mathbf{v}(t) \neq \mathbf{0}$.



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The rules for addition and scalar multiplication of vectors imply that

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \lim_{\Delta t \to 0} \left(\frac{x(t+\Delta t) - x(t)}{\Delta t}\mathbf{i} + \frac{y(t+\Delta t) - y(t)}{\Delta t}\mathbf{j} + \frac{z(t+\Delta t) - z(t)}{\Delta t}\mathbf{k} \right) \\ &= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.\end{aligned}$$

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Thus, the vector function $\mathbf{r}$ is differentiable at t if and only if its three scalar components, x, y, and z, are differentiable at t.

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Continuing our analysis of the moving particle, we define the **acceleration** of the particle to be the time derivative of the velocity:

$$\text{acceleration:} \quad \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}.$$

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which satisfy $y = x^2$ and $z = x^3$, the curve is the curve of intersection of the two cylinders $y = x^2$ and $z = x^3$.

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At any time t the velocity and acceleration vectors are given by

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The point $(1, 1, 1)$ on the curve corresponds to $t = 1$, so the velocity and acceleration at that point are $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{a} = 2\mathbf{j} + 6\mathbf{k}$, respectively.

THEOREM Differentiation rules for vector functions

Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be differentiable vector-valued functions, and let $\lambda(t)$ be a differentiable scalar-valued function. Then $\mathbf{u}(t) + \mathbf{v}(t)$, $\lambda(t)\mathbf{u}(t)$, $\mathbf{u}(t) \cdot \mathbf{v}(t)$, $\mathbf{u}(t) \times \mathbf{v}(t)$, and $\mathbf{u}(\lambda(t))$ are differentiable, and

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(e) $\dfrac{d}{dt}\Big(\mathbf{u}(\lambda(t))\Big) = \lambda'(t)\mathbf{u}'(\lambda(t))$

Also, at any point where $\mathbf{u}(t) \neq \mathbf{0}$,

(f) $\dfrac{d}{dt}|\mathbf{u}(t)| = \dfrac{\mathbf{u}(t) \cdot \mathbf{u}'(t)}{|\mathbf{u}(t)|}$

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\begin{aligned}
|\mathbf{v}| = \text{constant} &\iff |\mathbf{v}|^2 = \text{constant} \\
&\iff \mathbf{v}\cdot\mathbf{v} = \text{constant} \\
&\iff \frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = 0 \\
&\iff \frac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = 0 \\
&\iff 2\underbrace{\frac{d\mathbf{v}}{dt}}_{\mathbf{a}}\cdot\mathbf{v} = 0 \\
&\iff \mathbf{a}\perp\mathbf{v}
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$$\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \leq t \leq b.$$

The parameter t, here, need no longer represent time or any other specific quantity.

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Solution: We need to express all three coordinates of an arbitrary point on the line as functions of $t = y$. Since $y = t$, the equation $y = 2x - 4$ assures us that

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Since the line segment goes from $y = 0$ to $y = 2$, the required parametrization is

$$\mathbf{r} = \frac{t + 4}{2}\mathbf{i} + t\mathbf{j} + \left(\frac{3}{2}t + 7\right)\mathbf{k}, \quad 0 \le t \le 2.$$

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We could use $t = y$ instead of $t = x$ as the parameter; in this case the parametrization would be

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The plane $x + y = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Parametrize the whole parabola using $t = x$ as parameter. Could $t = y$ have been used as parameter? What about $t = z$?

Solution: From the equations of the two surfaces defining the parabola, we have $y = 1 - x = 1 - t$, and $z = x^2 + y^2 = 1 - 2t + 2t^2$. Thus the required parametrization is

$$\mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j} + (1 - 2t + 2t^2)\mathbf{k}, \quad -\infty < t < \infty.$$

We could use $t = y$ instead of $t = x$ as the parameter; in this case the parametrization would be

$$\mathbf{r} = (1-t)\mathbf{i} + t\mathbf{j} + (1 - 2t + t^2)\mathbf{k}, \quad -\infty < t < \infty.$$

What about $t = z$?

EXAMPLE
Show that each of the vector functions

$$\mathbf{r}_1(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}, \qquad (-\pi/2 \le t \le \pi/2),$$

$$\mathbf{r}_2(t) = (t-1)\mathbf{i} + \sqrt{2t - t^2}\,\mathbf{j}, \qquad (0 \le t \le 2), \quad \text{and}$$

$$\mathbf{r}_3(t) = t\sqrt{2 - t^2}\,\mathbf{i} + (1 - t^2)\mathbf{j}, \quad (-1 \le t \le 1)$$

all represent the same curve. Describe the curve.

Solution: $\mathbf{r}_1(-\pi/2) = -\mathbf{i}, \quad \mathbf{r}_1(\pi/2) = \mathbf{i}.$

The curve represented by $\mathbf{r}_1(t)$ lies in the upper half of the xy-plane since $\cos t \ge 0$ for all $-\pi/2 \le t \le \pi/2$. All points on the curve are at distance 1 from the origin:

$$|\mathbf{r}_1(t)| = \sqrt{(\sin t)^2 + (\cos t)^2} = 1.$$

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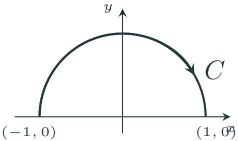


all represent the same curve. Describe the curve.

Solution: Therefore, $\mathbf{r}_1(t)$ represents the semicircle $y = \sqrt{1 - x^2}$ in the xy-plane traversed from left to right.

The other two functions have the same properties: both graphs lie in $y \ge 0$,

$\mathbf{r}_2(0) = -\mathbf{i}, \quad \mathbf{r}_2(2) = \mathbf{i},$ $\qquad$ $\mathbf{r}_3(-1) = -\mathbf{i}, \quad \mathbf{r}_3(1) = \mathbf{i},$

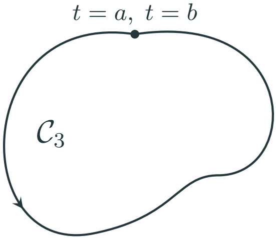
$|\mathbf{r}_2(t)| = \sqrt{(t-1)^2 + 2t - t^2} = 1.$ $\qquad$ $|\mathbf{r}_3(t)| = \sqrt{t^2(2 - t^2) + (1 - t^2)^2} = 1.$

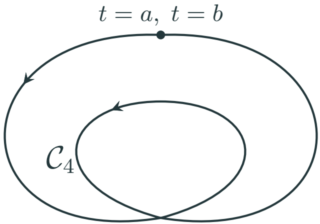
Thus, all three functions represent the same semicircle. Of course, the three parametrizations trace out the curve with different velocities.

Curves and Parametrizations

DEFINITION

The curve $\mathbf{r} = \mathbf{r}(t)$, $(a \leq t \leq b)$, is called a **closed curve** if $\mathbf{r}(a) = \mathbf{r}(b)$, that is, if the curve begins and ends at the same point.





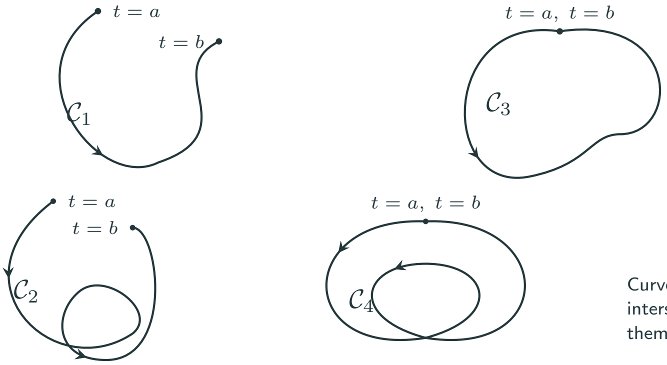
Curves and Parametrizations

The curve $\mathcal{C}$ is **non-self-intersecting** if there exists some parametrization

$$\mathbf{r} = \mathbf{r}(t),\ (a \leq t \leq b),$$

of $\mathcal{C}$ that is one-to-one except that the endpoints could be the same:

$$\mathbf{r}(t_1) = \mathbf{r}(t_2) \quad a \leq t_1 < t_2 \leq b \quad \implies t_1 = a \quad \text{and} \quad t_2 = b.$$



Curves $\mathcal{C}_2$ and $\mathcal{C}_4$ intersect themselves

Parametrizing the Curve of Intersection of Two Surfaces

EXAMPLE

Parametrize the curve of intersection of the plane $x + 2y + 4z = 4$ and the elliptic cylinder $x^2 + 4y^2 = 4$.

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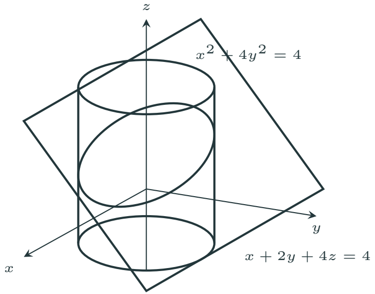
Thus, the given surfaces intersect in the curve

$$\mathbf{r} = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \left(1 - \frac{\cos t + \sin t}{2}\right)\mathbf{k},$$

$(0 \le t \le 2\pi)$.



Parametrizing the Curve of Intersection of Two Surfaces

EXAMPLE

Find a parametric representation of the curve of intersection of the two surfaces:

$$x^2 + y + z = 2 \quad \text{and} \quad xy + z = 1.$$

Solution:

- **Step 1: Eliminate one variable.** Subtracting the two equations to eliminate z:

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- **Step 3: Express z in terms of t.** $z = 1 - t(1+t) = 1 - t - t^2$
- **Final Result:** $\mathbf{r} = t\mathbf{i} + (1+t)\mathbf{j} + (1 - t - t^2)\mathbf{k}$

Arc-Length Parametrization

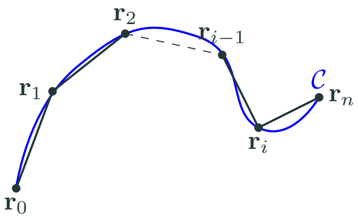
Curves and Parametrizations: Arc Length

Let $\mathcal{C}$ be a bounded, continuous curve specified by $\mathbf{r} = \mathbf{r}(t)$ for $a \le t \le b$.

Linear Approximation: Subdivide $[a, b]$ into n subintervals by points $a = t_0 < t_1 < \cdots < t_n = b$.

- **Polygonal Path Sum:**

$$s_n = \sum_{i=1}^{n} |\mathbf{r}_i - \mathbf{r}_{i-1}| = \sum_{i=1}^{n} \left| \frac{\Delta \mathbf{r}_i}{\Delta t_i} \right| \Delta t_i$$



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- **The Integral Limit:** As $n \to \infty$ and $\max \Delta t_i \to 0$:

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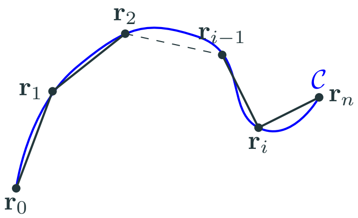

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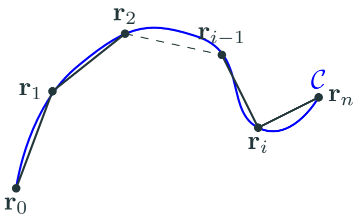


Arc Length Formula (in terms of speed)

$$s = \int_a^b |\mathbf{v}(t)| \, dt = \int_a^b v(t) \, dt$$

Example: Arc Length of a Circular Helix

EXAMPLE: Find the length s of the circular helix:

$$\mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$$

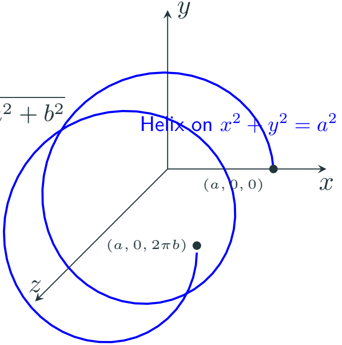
between the points $(a, 0, 0)$ and $(a, 0, 2\pi b)$.

Solution:

- **Step 1: Find Velocity and Speed.**

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}$$

$$v = |\mathbf{v}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2 + b^2}$$



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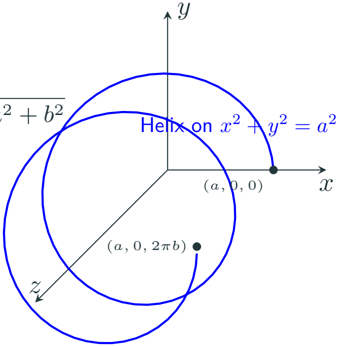

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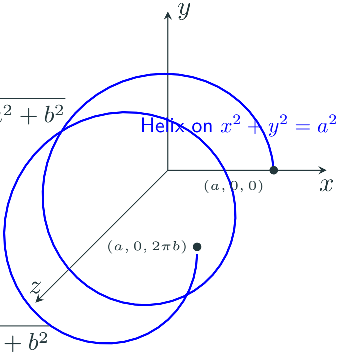

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- **Step 3: Integrate for length.**

$$s = \int_0^{2\pi} v(t)\,dt = \int_0^{2\pi} \sqrt{a^2 + b^2}\,dt = 2\pi\sqrt{a^2 + b^2}$$

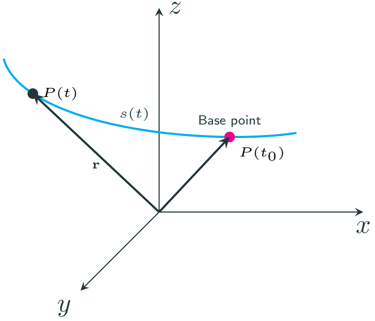
The Arc-Length Parametrization

Given a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ for $a \le t \le b$ and a fixed number $t_0 \in [a, b]$.

Definition:

We define the function $s(t)$ as the integral of speed from the base point:

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)| \, d\tau$$



Arc Length Parameter

s is called an **arc length parameter** for the curve.

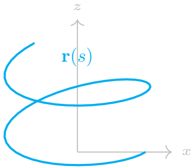
It represents the directed distance along the curve from the base point $P(t_0)$ to $P(t)$.

Arc-Length Parametrization: Circular Helix

EXAMPLE: Parametrize the circular helix $\mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$ in terms of the arc length s measured from $(a, 0, 0)$.

Solution:

- **Initial Point:** At $(a, 0, 0)$, the parameter is $t = 0$.



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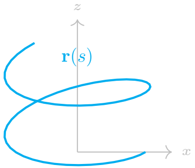

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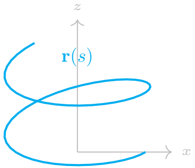
- **Invert for t:** Solve for the original parameter:

$$t = \frac{s}{\sqrt{a^2 + b^2}}$$



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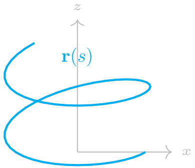

The Arc-Length Parametrization

$$\mathbf{r}(s) = a\cos\left(\frac{s}{\sqrt{a^2 + b^2}}\right)\mathbf{i} + a\sin\left(\frac{s}{\sqrt{a^2 + b^2}}\right)\mathbf{j} + \frac{bs}{\sqrt{a^2 + b^2}}\mathbf{k}$$