MAT124 MATHEMATICS II

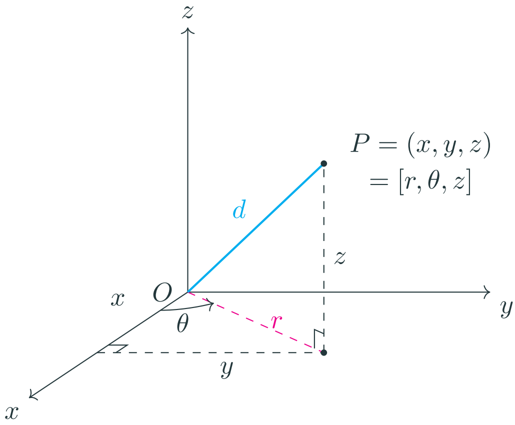
Cylindrical and Spherical Coordinates
&
Functions of Several Variables

Outline

Cylindrical Coordinates

Cylindrical and Spherical Coordinates

Cylindrical Coordinates

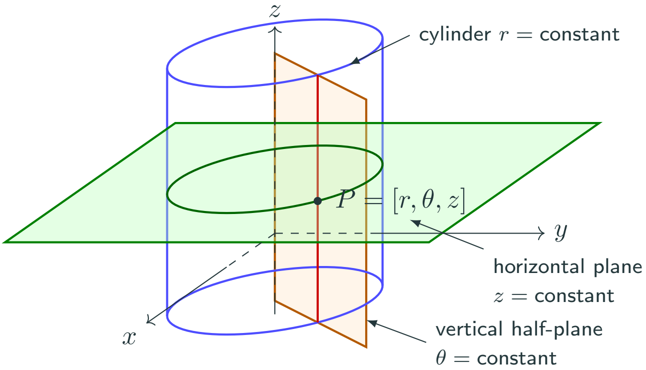


$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

$$d = \sqrt{r^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$$

Cylindrical Coordinates

The point with Cartesian coordinates $(1, 1, 1)$ has cylindrical coordinates $[\sqrt{2}, \pi/4, 1]$. The point with Cartesian coordinates $(0, 2, -3)$ has cylindrical coordinates $[2, \pi/2, -3]$. The point with cylindrical coordinates $[4, -\pi/3, 5]$ has Cartesian coordinates $(2, -2\sqrt{3}, 5)$.



Cylindrical Coordinates

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- The z-curves are intersections of the cylinders $r =$ constant and the half-planes $\theta =$ constant, and so are **vertical straight lines**.

Cylindrical Coordinates

EXAMPLE

Identify the surfaces whose equations in cylindrical coordinates are:

(a) $z = r^2$, (b) $z = r\cos\theta$, (c) $r = 2\cos\theta$.

Solution:

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Solution:

(a) $z = r^2$ represents the circular paraboloid with Cartesian equation $z = x^2 + y^2$. It has vertex at the origin and axis of symmetry along the positive z-axis.

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(c) $r = 2\cos\theta$ can be rewritten $r^2 = 2r\cos\theta$, so represents the vertical surface with Cartesian equation $x^2 + y^2 = 2x$. This is a circular cylinder of radius 1 with central axis along the vertical line through the point $(1, 0, 0)$ (in Cartesian coordinates).

Cylindrical Coordinates

EXAMPLE

Describe the curves whose equations in cylindrical coordinates are:

(a) $\begin{cases} r = z \\ z = 1 + r\cos\theta \end{cases}$

(b) $\begin{cases} \theta = \pi/2 \\ r^2 + z^2 = 4 \end{cases}$

Solution:

Cylindrical Coordinates

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Solution:

(a) The curve is the parabola in which the plane $z = 1 + x$ intersects the right-circular half-cone $z = \sqrt{x^2 + y^2}$. Since the plane is parallel to the line $z = x$, which is a generator of the cone, the intersection must be a parabola rather than an ellipse or a hyperbola.

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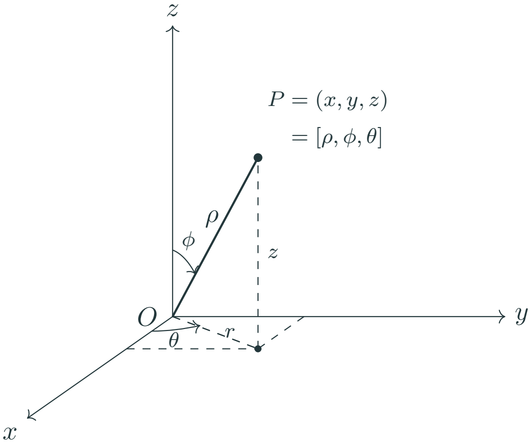
(b) $\theta = \pi/2$ represents the half of the yz-plane where $y \geq 0$. $r^2 + z^2 = 4$ represents a sphere of radius 2 centred at the origin. Thus, this curve is the semicircle with Cartesian equation $y = \sqrt{4 - z^2}$ in the plane $x = 0$.

Spherical Coordinates

Cylindrical and Spherical Coordinates

Spherical Coordinates



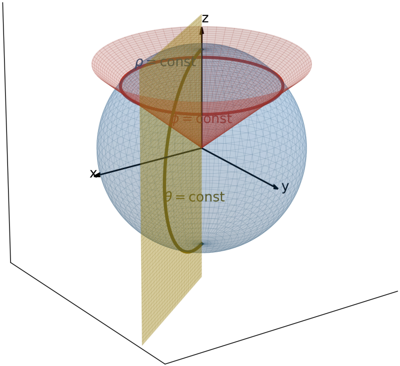
$$x = \overbrace{\rho \sin\phi}^{r} \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

Spherical Coordinates



$\rho =$ constant

$\phi =$ constant

$\theta =$ constant

Spherical Coordinates

EXAMPLE

Find:

(a) the Cartesian coordinates of the point P with spherical coordinates $[2, \pi/3, \pi/2]$, and

(b) the spherical coordinates of the point Q with Cartesian coordinates $(1, 1, \sqrt{2})$.

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(b) the spherical coordinates of the point Q with Cartesian coordinates $(1, 1, \sqrt{2})$.

Solution: (a) If $\rho = 2$, $\phi = \pi/3$, and $\theta = \pi/2$, then

$$x = 2\sin(\pi/3)\cos(\pi/2) = 0$$

$$y = 2\sin(\pi/3)\sin(\pi/2) = \sqrt{3}$$

$$z = 2\cos(\pi/3) = 1.$$

The Cartesian coordinates of P are $(0, \sqrt{3}, 1)$.

Spherical Coordinates

EXAMPLE

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(a) the Cartesian coordinates of the point P with spherical coordinates $[2, \pi/3, \pi/2]$, and

(b) the spherical coordinates of the point Q with Cartesian coordinates $(1, 1, \sqrt{2})$.

Solution: (b) Given that

$$\rho \sin\phi \cos\theta = x = 1$$
$$\rho \sin\phi \sin\theta = y = 1$$
$$\rho \cos\phi = z = \sqrt{2},$$

we calculate that $\rho^2 = 1 + 1 + 2 = 4$, so $\rho = 2$. Also $r^2 = 1 + 1 = 2$, so $r = \sqrt{2}$. Thus, $\tan\phi = r/z = 1$, so $\phi = \pi/4$. Also, $\tan\theta = y/x = 1$, so $\theta = \pi/4$ or $5\pi/4$. Since $x > 0$, we must have $\theta = \pi/4$. The spherical coordinates of Q are $[2, \pi/4, \pi/4]$.

Partial Differentiation

A **function** f of n real variables is a rule that assigns a *unique* real number $f(x_1, x_2, \ldots, x_n)$ to each point $(x_1, x_2, \ldots, x_n)$ in some subset $\mathcal{D}(f)$ of $\mathbb{R}^n$. $\mathcal{D}(f)$ is called the **domain** of f. The set of real numbers $f(x_1, x_2, \ldots, x_n)$ obtained from points in the domain is called the **range** of f.

Functions of Several Variables

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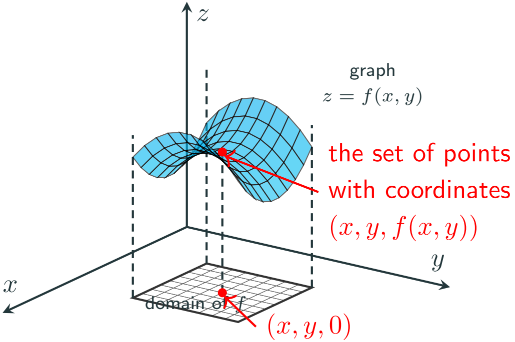
As for functions of one variable, the **domain convention** specifies that the domain of a function of n variables is the largest set of points $(x_1, x_2, \ldots, x_n)$ for which $f(x_1, x_2, \ldots, x_n)$ makes sense as a real number, unless that domain is explicitly stated to be a smaller set.

Graphs



The graph of $f(x, y)$ is the surface with equation $z = f(x, y)$ defined for points (x, y) in the domain of f

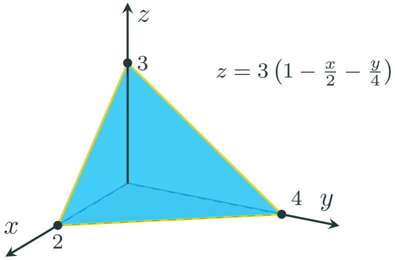
Graphs

EXAMPLE

Consider the function

$$f(x, y) = 3\left(1 - \frac{x}{2} - \frac{y}{4}\right), \qquad (0 \le x \le 2, \quad 0 \le y \le 4 - 2x).$$

The graph of f is the plane triangular surface with vertices at $(2, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 3)$.

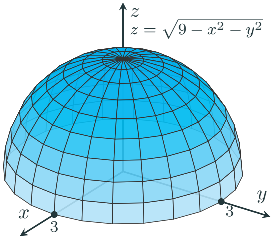


Graphs

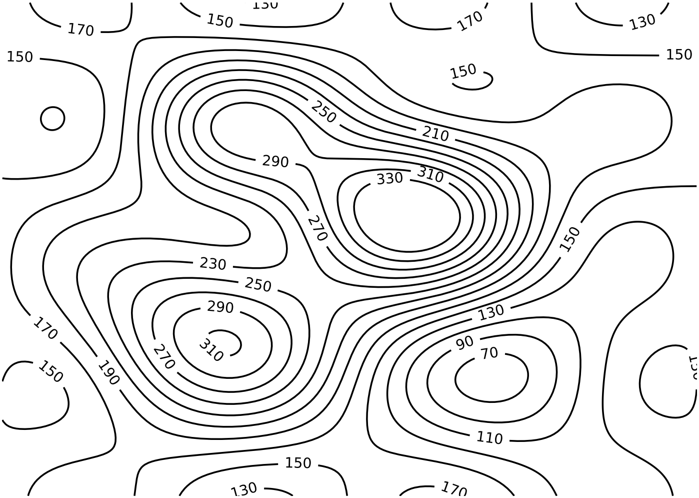
EXAMPLE

Consider $f(x, y) = \sqrt{9 - x^2 - y^2}$. The expression under the square root cannot be negative, so the domain is the disk $x^2 + y^2 \leq 9$ in the xy-plane.

If we square the equation $z = \sqrt{9 - x^2 - y^2}$, we can rewrite the result in the form $x^2 + y^2 + z^2 = 9$. This is a sphere of radius 3 centred at the origin. However, the graph of f is only the upper hemisphere where $z \geq 0$.

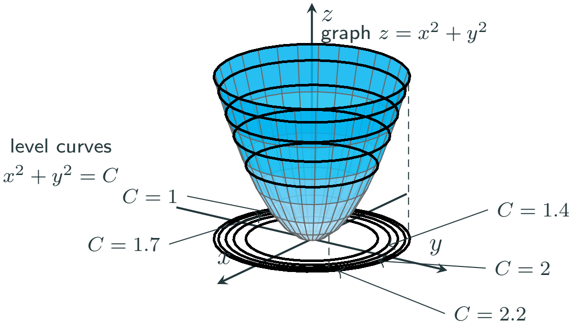


Level Curves



Level Curves

In the xy-plane we sketch the curves $f(x, y) = C$ for various values of the constant C. These curves are called **level curves** of f because they are the vertical projections onto the xy-plane of the curves in which the graph $z = f(x, y)$ intersects the horizontal (level) planes $z = C$.



Level Curves

EXAMPLE

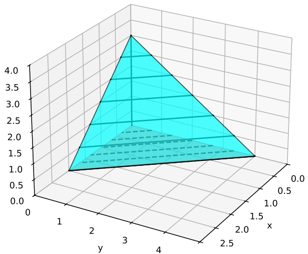
The level curves of the function

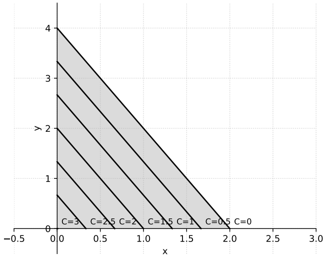
$$f(x,y) = 3\left(1 - \frac{x}{2} - \frac{y}{4}\right), \qquad (0 \leq x \leq 2, \quad 0 \leq y \leq 4 - 2x)$$

are the segments of the straight lines

$$3\left(1 - \frac{x}{2} - \frac{y}{4}\right) = C \qquad \text{or} \qquad \frac{x}{2} + \frac{y}{4} = 1 - \frac{C}{3}, \qquad (0 \leq C \leq 3),$$

which lie in the first quadrant.



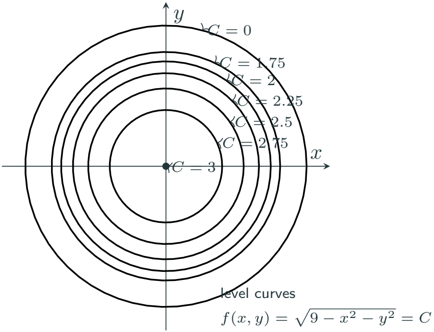


Level Curves

EXAMPLE

The level curves of the function $f(x, y) = \sqrt{9 - x^2 - y^2}$ are the concentric circles

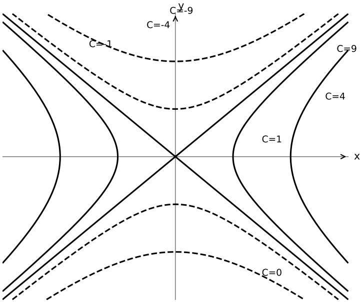
$$\sqrt{9 - x^2 - y^2} = C \qquad \text{or} \qquad x^2 + y^2 = 9 - C^2, \qquad (0 \leq C \leq 3).$$

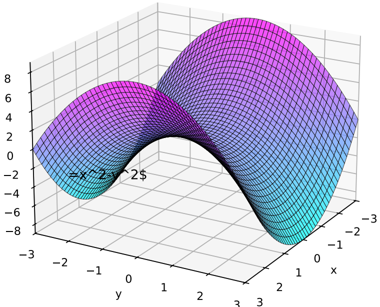


Level Curves

EXAMPLE

The level curves of the function $f(x,y) = x^2 - y^2$ are the curves $x^2 - y^2 = C$. For $C = 0$ the level "curve" is the pair of straight lines $x = y$ and $x = -y$. For other values of C the level curves are rectangular hyperbolas with these lines as asymptotes.



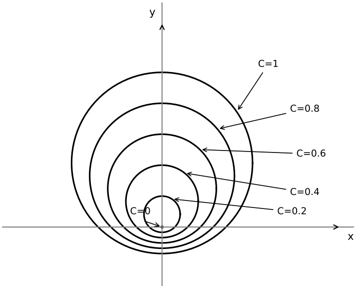


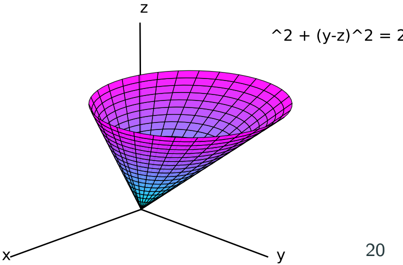
Level Curves

EXAMPLE

Describe and sketch some level curves of the function $z = g(x, y)$ defined by $z \geq 0$, and $x^2 + (y - z)^2 = 2z^2$. Also sketch the graph of the function g.

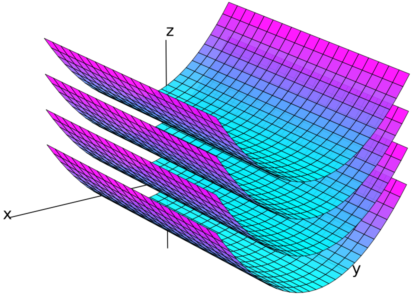
Solution The level curve $z = g(x, y) = C$ (where C is a positive constant) has equation $x^2 + (y - C)^2 = 2C^2$ and is. therefore. a circle of radius $\sqrt{2}C$ centred at $(0, C)$.





Level Curves

For a function $f(x, y, z)$ of three variables, the graph of the equation $f(x, y, z) = C$, where C is a constant is called a **level surface** of the function $f(x, y, z)$.



Level surfaces of

$$f(x, y, z) = x^2 - z$$