MAT124 MATHEMATICS II

Chain Rule, Gradients and Directional Derivatives

Outline

Chain Rule

Linear Approximations, Differentiability and Differentials

Gradients and Directional Derivatives

Chain Rule

The Chain Rule

$$z = f\big(u(t),\, v(t)\big) = g(t),$$

$$g'(t) = \lim_{h\to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h\to 0} \frac{f\big(u(t+h), v(t+h)\big) - f\big(u(t), v(t)\big)}{h}$$

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$$= \lim_{h\to 0} \frac{f(u(t+h), v(t+h)) - f(u(t), v(t+h))}{h} + \lim_{h\to 0} \frac{f(u(t), v(t+h)) - f(u(t), v(t))}{h}$$

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$$\begin{aligned} &= \lim_{h\to 0} \frac{f(u(t+h), v(t+h)) - f(u(t), v(t+h))}{h} \\ &\quad + \lim_{h\to 0} \frac{f(u(t), v(t+h)) - f(u(t), v(t))}{h} \end{aligned}$$

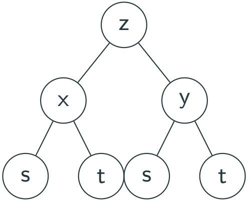
$$= f_1\big(u(t), v(t)\big)\, u'(t) + f_2\big(u(t), v(t)\big)\, v'(t)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

The Chain Rule

If z is a function of x and y with continuous first partial derivatives, and if x and y depend on s and t, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s},$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$



The Chain Rule

EXAMPLE:

If $z = \sin(x^2y)$, where $x = st^2$ and $y = s^2 + \dfrac{1}{t}$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$

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$$\begin{aligned}
\frac{\partial z}{\partial s} &= (2xy\cos(x^2y))t^2 + (x^2\cos(x^2y))2s \\
&= \left(2st^2\left(s^2 + \frac{1}{t}\right)t^2 + 2s^3t^4\right)\cos(s^4t^4 + s^2t^3) \\
&= (4s^3t^4 + 2st^3)\cos(s^4t^4 + s^2t^3)
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\frac{\partial z}{\partial t} &= (2xy\cos(x^2 y))2st + (x^2\cos(x^2 y))\left(-\frac{1}{t^2}\right) \\
&= \left(2st^2\left(s^2 + \frac{1}{t}\right)2st + s^2t^4\left(-\frac{1}{t^2}\right)\right)\cos(s^4t^4 + s^2t^3) \\
&= (4s^4t^3 + 3s^2t^2)\cos(s^4t^4 + s^2t^3).
\end{aligned}$$

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EXAMPLE:

Find $\dfrac{\partial}{\partial x} f(x^2 y, x+2y)$ and $\dfrac{\partial}{\partial y} f(x^2 y, x+2y)$ in terms of the partial derivatives of f, assuming that these partial derivatives are continuous.

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$$\begin{aligned}
&\frac{\partial}{\partial x} f(x^2 y, x + 2y) \\
&\qquad = f_1(x^2 y, x + 2y)\frac{\partial}{\partial x}(x^2 y) + f_2(x^2 y, x + 2y)\frac{\partial}{\partial x}(x + 2y) \\
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&\qquad = f_1(x^2y, x+2y)\frac{\partial}{\partial y}(x^2y) + f_2(x^2y, x+2y)\frac{\partial}{\partial y}(x+2y)\\
&\qquad = x^2f_1(x^2y, x+2y) + 2f_2(x^2y, x+2y).
\end{aligned}$$

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Express the partial derivatives of $z = h(s,t) = f(g(s,t))$ in terms of the derivative f' of f and the partial derivatives of g.

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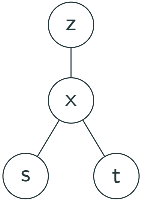

$$h_1(s,t) = \frac{\partial z}{\partial s} = \frac{dz}{dx}\frac{\partial x}{\partial s} = f'(g(s,t))g_1(s,t),$$
$$h_2(s,t) = \frac{\partial z}{\partial t} = \frac{dz}{dx}\frac{\partial x}{\partial t} = f'(g(s,t))g_2(s,t).$$

The Chain Rule

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Write the appropriate version of the Chain Rule for $\dfrac{\partial z}{\partial x}$, where z depends on u, v, and r; u and v depend on x, y, and r; and r depends on x and y.

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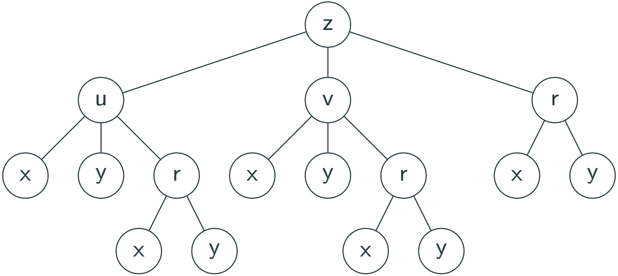

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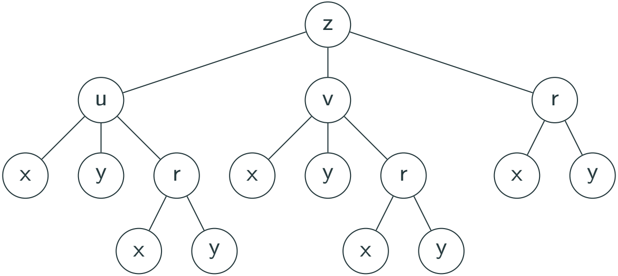

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial r}\frac{\partial r}{\partial x}.$$

The Chain Rule

Homogeneous Functions

A function $f(x_1, \dots, x_n)$ is said to be positively homogeneous of degree k if, for every point $(x_1, x_2, \dots, x_n)$ in its domain and every real number $t > 0$, we have

$$f(tx_1, tx_2, \dots, tx_n) = t^k\, f(x_1, \dots, x_n).$$

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EXAMPLES

- $f(x, y) = x^2 + xy - y^2$ is positively homogeneous of degree 2,

The Chain Rule

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- $f(x, y) = \dfrac{2xy}{x^2 + y^2}$ is positively homogeneous of degree 0
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- $f(x, y, z) = \dfrac{x - y + 5z}{yz - z^2}$ is positively homogeneous of degree -1,
- $f(x, y) = x^2 + y$ is not positively homogeneous.

The Chain Rule

Homogeneous Functions

Euler's Theorem

If $f(x_1, \ldots, x_n)$ has continuous first partial derivatives and is positively homogeneous of degree k, then

$$\sum_{i=1}^{n} x_i f_i(x_1, \ldots, x_n) = k f(x_1, \ldots, x_n).$$

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$$\sum_{i=1}^{n} x_i f_i(x_1, \ldots, x_n) = k f(x_1, \ldots, x_n).$$

Proof: Differentiate the equation $f(tx_1, tx_2, \ldots, tx_n) = t^k f(x_1, \ldots, x_n)$ with respect to t to get

$$\begin{aligned} &x_1 f_1(tx_1, \ldots, tx_n) + x_2 f_2(tx_1, \ldots, tx_n) + \cdots + x_n f_n(tx_1, \ldots, tx_n) \\ &\quad = k t^{k-1} f(x_1, \ldots, x_n). \end{aligned}$$

Now substitute $t = 1$ to get the desired result.

The Chain Rule

Second order partial derivatives

EXAMPLE:

Calculate $\dfrac{\partial^2}{\partial x \partial y} f(x^2 - y^2, xy)$ in terms of partial derivatives of the function f. Assume that the second-order partials of f are continuous.

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Solution:

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$$\frac{\partial}{\partial y} f(u, v) = -2y f_1(u, v) + x f_2(u, v).$$

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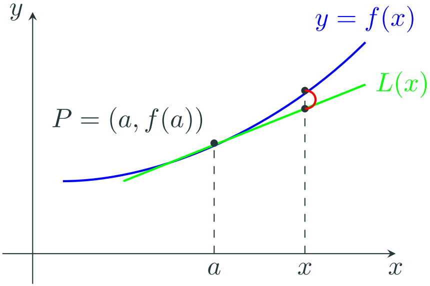
$$\frac{\partial}{\partial y} f(u, v) = -2y f_1(u, v) + x f_2(u, v).$$

$$\begin{aligned}\frac{\partial^2}{\partial x \partial y} f(u, v) &= -2y\left(2x f_{11}(u, v) + y f_{12}(u, v)\right) \\ &\quad + f_2(u, v) + x\left(2x f_{21}(u, v) + y f_{22}(u, v)\right) \\ &= f_2(u, v) - 4xy f_{11}(u, v) + 2(x^2 - y^2) f_{12}(u, v) + xy f_{22}(u, v).\end{aligned}$$

Linear Approximations, Differentiability and Differentials

Linear Approximations



$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

Here, $L(x)$ is the linearization of f at a; its graph is the tangent line to $y = f(x)$ at $x = a$.

Linear Approximations

Similarly, the tangent plane to the graph of $z = f(x, y)$ at (a, b) is

$$z = L(x, y) = f(a, b) + f_1(a, b)(x - a) + f_2(a, b)(y - b)$$

which is the linearization of f at (a, b). We can use $L(x, y)$ to approximate values of $f(x, y)$ near (a, b):

$$f(x, y) \approx L(x, y) = f(a, b) + f_1(a, b)(x - a) + f_2(a, b)(y - b).$$

Linear Approximations

EXAMPLE

Find an approximate value for $f(x,y) = \sqrt{2x^2 + e^{2y}}$ at $(2.2, -0.2)$.

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Solution: It is convenient to use the linearization at $(2, 0)$, where the values of f and its partials are easily evaluated:

$$f_1(x, y) = \frac{2x}{\sqrt{2x^2 + e^{2y}}},$$

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$$f_1(x,y) = \frac{2x}{\sqrt{2x^2 + e^{2y}}}, \qquad f(2,0) = 3,$$

$$f_1(2,0) = \frac{4}{3},$$

$$f_2(x,y) = \frac{e^{2y}}{\sqrt{2x^2 + e^{2y}}}, \qquad f_2(2,0) = \frac{1}{3}.$$

Linear Approximations

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Find an approximate value for $f(x,y)=\sqrt{2x^2+e^{2y}}$ at $(2.2,-0.2)$.

Solution: It is convenient to use the linearization at $(2,0)$, where the values of f and its partials are easily evaluated:

$$f_1(x,y)=\frac{2x}{\sqrt{2x^2+e^{2y}}}, \qquad f(2,0)=3,$$

$$f_2(x,y)=\frac{e^{2y}}{\sqrt{2x^2+e^{2y}}}, \qquad f_1(2,0)=\frac{4}{3},$$

$$f_2(2,0)=\frac{1}{3}.$$

Thus,

$$L(x,y)=3+\frac{4}{3}(x-2)+\frac{1}{3}(y-0)$$

and

$$f(2.2,-0.2)\approx L(2.2,-0.2)=3+\frac{4}{3}(2.2-2)+\frac{1}{3}(-0.2-0)=3.2.$$

Differentiability

Definition

We say that the function $f(x, y)$ is differentiable at the point (a, b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a,b) - hf_1(a,b) - kf_2(a,b)}{\sqrt{h^2+k^2}} = 0.$$

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The function $f(x,y)$ is differentiable at the point (a,b) if and only if the surface $z=f(x,y)$ has a *nonvertical tangent plane* at (a,b). This implies that $f_1(a,b)$ and $f_2(a,b)$ must exist and that f must be continuous at (a,b).

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differentiability $\Rightarrow$ *continuity*

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Theorem

If f_1 and f_2 are continuous in a neighbourhood of the point (a, b), then f is differentiable at (a, b).

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Calculate $f(x+h, y+k) - f(x, y) - f_1(x, y)h - f_2(x, y)k$ if $f(x, y) = x^3 + xy^2$.

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Solution: Since $f_1(x, y) = 3x^2 + y^2$ and $f_2(x, y) = 2xy$, we have

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$$\begin{aligned}
&f(x+h, y+k) - f(x, y) - f_1(x, y)h - f_2(x, y)k \\
&= (x+h)^3 + (x+h)(y+k)^2 - x^3 - xy^2 - (3x^2 + y^2)h - 2xyk \\
&= 3xh^2 + h^3 + 2yhk + hk^2 + xk^2.
\end{aligned}$$

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Solution: Since $f_1(x, y) = 3x^2 + y^2$ and $f_2(x, y) = 2xy$, we have

$$\begin{aligned} & f(x+h, y+k) - f(x, y) - f_1(x, y)h - f_2(x, y)k \\ & = (x+h)^3 + (x+h)(y+k)^2 - x^3 - xy^2 - (3x^2 + y^2)h - 2xyk \\ & = 3xh^2 + h^3 + 2yhk + hk^2 + xk^2. \end{aligned}$$

$$\lim_{(h,k)\to(0,0)} \frac{3xh^2 + h^3 + 2yhk + hk^2 + xk^2}{\sqrt{h^2 + k^2}} = 0.$$

Differentials

If the first partial derivatives of a function $z = f(x_1, \ldots, x_n)$ exist at a point, we may construct a differential dz or df of the function at that point in a manner similar to that used for functions of one variable:

$$\begin{aligned} dz = df &= \frac{\partial z}{\partial x_1} dx_1 + \frac{\partial z}{\partial x_2} dx_2 + \cdots + \frac{\partial z}{\partial x_n} dx_n \\ &= f_1(x_1, \ldots, x_n) dx_1 + \cdots + f_n(x_1, \ldots, x_n) dx_n. \end{aligned}$$

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For a *differentiable* function f, the differential df is an approximation to the change Δf in value of the function given by

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Differentials

EXAMPLE

Estimate the percentage change in the period $T = 2\pi\sqrt{\frac{L}{g}}$ of a simple pendulum if the length, L, of the pendulum increases by 2% and the acceleration of gravity, g, decreases by 0.6%.

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Solution: We calculate the differential of T:

$$dT = \frac{\partial T}{\partial L}dL + \frac{\partial T}{\partial g}dg = \frac{2\pi}{2\sqrt{Lg}}dL - \frac{2\pi\sqrt{L}}{2g^{3/2}}dg.$$

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We are given that $dL = \frac{2}{100}L$ and $dg = -\frac{6}{1{,}000}g$. Thus,

$$dT = \frac{1}{100}2\pi\sqrt{\frac{L}{g}} - \left(-\frac{6}{1{,}000}\right)2\pi\frac{1}{2}\sqrt{\frac{L}{g}} = \frac{13}{1{,}000}T.$$

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Therefore, the period T of the pendulum increases by 1.3%.

Gradients and Directional Derivatives

Gradients

Gradient

At any point (x, y) where the first partial derivatives of the function $f(x, y)$ exist, we define the gradient vector $\nabla f(x, y) = \mathbf{grad}\, f(x, y)$ by

$$\nabla f(x, y) = \mathbf{grad}\, f(x, y) = f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}.$$

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For the basis vector $\mathbf{i}$ and $\mathbf{j}$ in $\mathbb{R}^2$, the symbol ∇, called del or nabla, is a *vector differential operator*:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}.$$

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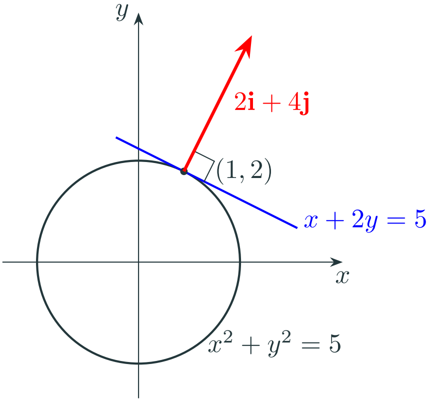
We can *apply* this operator to a function $f(x, y)$ by writing the operator to the left of the function. The result is the gradient of the function

$$\nabla f(x, y) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right) f(x, y) = f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}.$$

Gradient

EXAMPLE

If $f(x,y) = x^2 + y^2$, then $\nabla f(x,y) = 2x\mathbf{i} + 2y\mathbf{j}$. In particular, $\nabla f(1,2) = 2\mathbf{i} + 4\mathbf{j}$. Observe that this vector is perpendicular to the tangent line $x + 2y = 5$ to the circle $x^2 + y^2 = 5$ at $(1,2)$. This circle is the level curve of f that passes through the point $(1,2)$.



Gradient

Theorem

If $f(x, y)$ is differentiable at the point (a, b) and $\nabla f(a, b) \neq 0$, then $\nabla f(a, b)$ is a normal vector to the level curve of f that passes through (a, b).

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Proof: Let $\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ be a parametrization of the level curve of f such that $x(0) = a$ and $y(0) = b$. Then for all t near 0, $f\big(x(t), y(t)\big) = f(a, b)$. Differentiating this equation with respect to t using the Chain Rule, we obtain

$$f_1(x(t), y(t))\,\frac{dx}{dt} + f_2(x(t), y(t))\,\frac{dy}{dt} = 0.$$

At $t = 0$ this says that

$$\nabla f(a, b) \cdot \left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = 0;$$

that is, ∇f is perpendicular to the tangent vector $d\mathbf{r}/dt$ to the level curve at (a, b).

Directional Derivative

Directional Derivative

Let $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ be a unit vector, so that $u^2 + v^2 = 1$. The directional derivative of $f(x, y)$ at (a, b) in the direction of $\mathbf{u}$ is the rate of change of $f(x, y)$ with respect to distance measured at (a, b) along a ray in the direction of $\mathbf{u}$ in the xy-plane. This directional derivative is given by

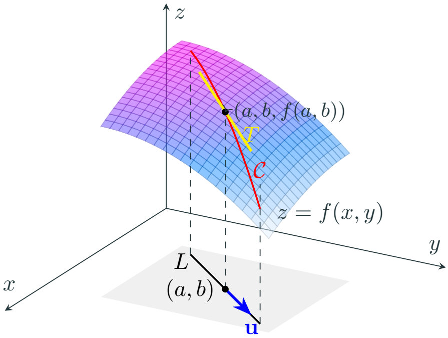
$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0^+} \frac{f(a + hu,\ b + hv) - f(a, b)}{h}.$$

It is also given by

$$D_{\mathbf{u}} f(a, b) = \frac{d}{dt} f(a + tu,\ b + tv)\Big|_{t=0}$$

if the derivative on the right–hand side exists.

Directional Derivative



Unit vector $\mathbf{u}$ determines a line L through (a, b) in the domain of f. The vertical plane containing L intersects the graph of f in a curve C whose tangent T at $(a, b, f(a, b))$ has slope $D_{\mathbf{u}} f(a, b)$.

Directional Derivative

Theorem - Using the gradient to find directional derivatives

If f is differentiable at (a, b) and $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ is a unit vector, then the directional derivative of f at (a, b) in the direction of $\mathbf{u}$ is given by

$$D_{\mathbf{u}} f(a, b) = \mathbf{u} \cdot \nabla f(a, b).$$

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Proof: By the Chain Rule:

$$\begin{aligned} D_{\mathbf{u}}f(a,b) &= \left.\frac{d}{dt}f(a+tu, b+tv)\right|_{t=0} \\ &= uf_1(a,b) + vf_2(a,b) = \mathbf{u} \cdot \nabla f(a,b). \end{aligned}$$

Directional Derivative

Given any nonzero vector $\mathbf{v}$, we can always obtain a unit vector in the same direction by dividing $\mathbf{v}$ by its length. The directional derivative of f at (a, b) in the direction of $\mathbf{v}$ is therefore given by

$$D_{\mathbf{v}/|\mathbf{v}|}f(a,b) = \frac{\mathbf{v}}{|\mathbf{v}|} \cdot \nabla f(a,b).$$

Directional Derivative Examples

EXAMPLE

Find the rate of change of $f(x, y) = y^4 + 2xy^3 + x^2y^2$ at $(0, 1)$ measured in each of the following directions:

(a) $\mathbf{i} + 2\mathbf{j}$, (b) $\mathbf{j} - 2\mathbf{i}$, (c) $3\mathbf{i}$, (d) $\mathbf{i} + \mathbf{j}$.

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Solution: We calculate

$$\begin{aligned}\nabla f(x, y) &= (2y^3 + 2xy^2)\mathbf{i} + (4y^3 + 6xy^2 + 2x^2y)\mathbf{j},\\ \nabla f(0, 1) &= 2\mathbf{i} + 4\mathbf{j}.\end{aligned}$$

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(a) The unit vector in the direction of $\mathbf{i} + 2\mathbf{j}$ is $\dfrac{\mathbf{i} + 2\mathbf{j}}{\sqrt{5}}$. Thus, the directional derivative of f at $(0, 1)$ in that direction is

$$\frac{\mathbf{i} + 2\mathbf{j}}{\sqrt{5}} \cdot (2\mathbf{i} + 4\mathbf{j}) = \frac{2 + 8}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}.$$

Observe that $\mathbf{i} + 2\mathbf{j}$ points in the same direction as $\nabla f(0, 1)$ so the directional derivative is positive and equal to the length of $\nabla f(0, 1)$.

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Solution: (b) The unit vector in the direction of $\mathbf{j} - 2\mathbf{i}$ is $\dfrac{\mathbf{j} - 2\mathbf{i}}{\sqrt{5}}$. Thus, the directional derivative of f at $(0, 1)$ in that direction is

$$\frac{-2\mathbf{i} + \mathbf{j}}{\sqrt{5}} \cdot (2\mathbf{i} + 4\mathbf{j}) = \frac{-4 + 4}{\sqrt{5}} = 0.$$

Since $\mathbf{j} - 2\mathbf{i}$ is perpendicular to $\nabla f(0, 1)$, it is tangent to the level curve of f through $(0, 1)$, so the directional derivative in that direction is zero.

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Solution: (c) The unit vector in the direction of $3\mathbf{i}$ is just $\mathbf{i}$, so the directional derivative of f at $(0, 1)$ in that direction is

$$\mathbf{i} \cdot (2\mathbf{i} + 4\mathbf{j}) = 2.$$

As noted previously, the directional derivative of f in the direction of the positive x-axis is just $f_1(0, 1)$.

Directional Derivative Examples

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Find the rate of change of $f(x, y) = y^4 + 2xy^3 + x^2y^2$ at $(0, 1)$ measured in each of the following directions:

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Solution: (d) The unit vector in the direction of $\mathbf{i} + \mathbf{j}$ is $\dfrac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$, so the directional derivative of f at $(0, 1)$ in that direction is

$$\frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} \cdot (2\mathbf{i} + 4\mathbf{j}) = \frac{2 + 4}{\sqrt{2}} = 3\sqrt{2}.$$

If we move along the surface $z = f(x, y)$ through the point $(0, 1, 1)$ in a direction making horizontal angles of 45° with the positive directions of the x- and y-axes, we would be rising at a rate of $3\sqrt{2}$ vertical units per horizontal unit moved.

Directional Derivative

For any unit vector $\mathbf{u}$ we have

$$D_{\mathbf{u}}f(a,b) = \mathbf{u} \cdot \nabla f(a,b) = |\nabla f(a,b)| \cos\theta,$$

where θ is the angle between the vectors $\mathbf{u}$ and $\nabla f(a,b)$.

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- The directional derivative is zero in the direction $\theta = \pi/2$; this is the direction of the (tangent line to the) level curve of f through (a, b).

Geometric Properties of the Gradient Vector

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- The rate of change of $f(x, y)$ at (a, b) is zero in directions tangent to the level curve of f that passes through (a, b).

Directional Derivative Example

EXAMPLE

The temperature at position (x, y) in a region of the xy-plane is $T\,^\circ$C, where

$$T(x, y) = x^2 e^{-y}.$$

In what direction at the point $(2, 1)$ does the temperature increase most rapidly? What is the rate of increase of f in that direction?

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Solution: We have

$$\nabla T(x, y) = 2xe^{-y}\mathbf{i} - x^2 e^{-y}\mathbf{j},$$

$$\nabla T(2, 1) = \frac{4}{e}\mathbf{i} - \frac{4}{e}\mathbf{j} = \frac{4}{e}(\mathbf{i} - \mathbf{j}).$$

At $(2, 1)$, $T(x, y)$ increases most rapidly in the direction of the vector $\mathbf{i} - \mathbf{j}$. The rate of increase in this direction is $|\nabla T(2, 1)| = 4\sqrt{2}/e$ °C/unit distance.

Directional Derivative Example

EXAMPLE

A hiker is standing beside a stream on the side of a mountain, examining her map of the region. The height of land (in metres) at any point (x, y) is given by the function

$$h(x, y) = \frac{20,000}{3 + x^2 + 2y^2},$$

where x and y (in kilometres) denote the coordinates of the point on the hiker's map. The hiker is at the point $(3, 2)$.

(a) What is the direction of flow of the stream at $(3, 2)$ on the hiker's map? How fast is the stream descending at her location?

(b) Find the equation of the path of the stream on the hiker's map.

(c) At what angle to the path of the stream (on the map) should the hiker set out if she wishes to climb at a 15° inclination to the horizontal?

(d) Make a sketch of the hiker's map, showing some curves of constant elevation, and showing the stream.

Directional Derivative Example

Solution: (a) We begin by calculating the gradient of h and its length at $(3, 2)$:

$$\begin{aligned} \nabla h(x, y) &= -\frac{20,000}{(3 + x^2 + 2y^2)^2}(2x\mathbf{i} + 4y\mathbf{j}), \\ \nabla h(3, 2) &= -100(3\mathbf{i} + 4\mathbf{j}), \\ |\nabla h(3, 2)| &= 500. \end{aligned}$$

The stream is flowing in the direction whose horizontal projection at $(3, 2)$ is $-\nabla h(3, 2)$, that is, in the horizontal direction of the vector $3\mathbf{i} + 4\mathbf{j}$. The stream is descending at a rate of 500 m/km, that is, 0.5 m per horizontal metre travelled.

Directional Derivative Example

Solution: (b) If the vector $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$ is tangent to the path of the stream at point (x, y) on the map, then $d\mathbf{r}$ is parallel to $\nabla h(x, y)$. Hence, the components of these two vectors are proportional:

$$\frac{dx}{2x} = \frac{dy}{4y} \quad \text{or} \quad \frac{dy}{y} = \frac{2dx}{x}.$$

Integrating both sides of this equation, we get $\ln y = 2 \ln x + \ln C$, or $y = Cx^2$. Since the path of the stream passes through $(3, 2)$, we have $C = 2/9$ and the equation is $9y = 2x^2$.

Directional Derivative Example

Solution: (c) Suppose the hiker moves away from $(3,2)$ in the direction of the unit vector $\mathbf{u}$. She will be ascending at an inclination of 15° if the directional derivative of h in the direction of $\mathbf{u}$ is $1,000 \tan 15^\circ \approx 268$. (The $1,000$ compensates for the fact that the vertical units are metres while the horizontal units are kilometres.) If θ is the angle between $\mathbf{u}$ and the upstream direction, then

$$500 \cos\theta = |\nabla h(3,2)| \cos\theta = D_{\mathbf{u}} h(3,2) \approx 268.$$

Hence, $\cos\theta \approx 0.536$ and $\theta \approx 57.6^\circ$. She should set out in a direction making a horizontal angle of about 58° with the upstream direction.

Directional Derivative Example

Solution: (d)