MAT124 MATHEMATICS II

Gradients and Directional Derivatives, Tangent Planes and Normal Lines, Implicit Functions

Outline

Gradients and Directional Derivatives

The Gradient in Three and More Dimensions

By analogy with the two-dimensional case, a function $f(x_1, x_2, \ldots, x_n)$ of n variables possessing first partial derivatives has gradient given by

$$\nabla f(x_1, x_2, \ldots, x_n) = \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \frac{\partial f}{\partial x_2}\mathbf{e}_2 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e}_n,$$

where $\mathbf{e}_j$ is the unit vector from the origin to the unit point on the jth coordinate axis.

The Gradient in Three and More Dimensions

By analogy with the two-dimensional case, a function $f(x_1, x_2, \dots, x_n)$ of n variables possessing first partial derivatives has gradient given by

$$\nabla f(x_1, x_2, \dots, x_n) = \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \frac{\partial f}{\partial x_2}\mathbf{e}_2 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e}_n,$$

where $\mathbf{e}_j$ is the unit vector from the origin to the unit point on the jth coordinate axis.

In particular, for a function of three variables,

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Gradients and Tangents to Level Curves

If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j}$ (making the curve a level curve of f), then $f(g(t), h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the equations

$$\frac{d}{dt} f(g(t), h(t)) = \frac{d}{dt}(c) \xrightarrow{\text{Chain Rule}} \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0$$

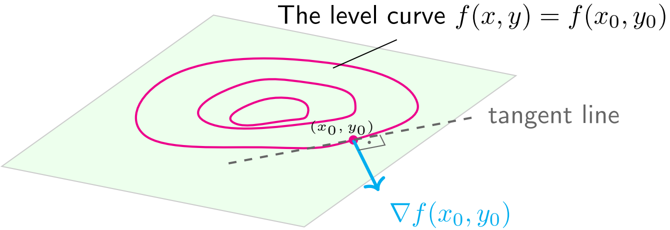
Gradients and Tangents to Level Curves

If a differentiable function $f(x,y)$ has a constant value c along a smooth curve $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j}$ (making the curve a level curve of f), then $f(g(t), h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the equations

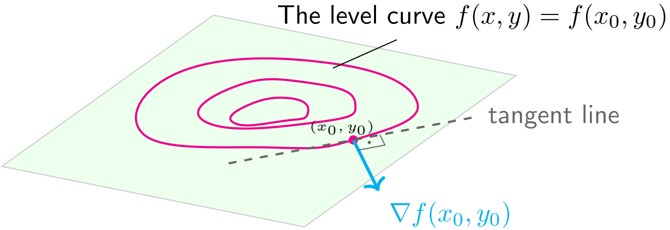
$$\frac{d}{dt} f(g(t), h(t)) = \frac{d}{dt}(c) \xrightarrow{\text{Chain Rule}} \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j}\right)}_{\frac{d\mathbf{r}}{dt}} = 0.$$

Gradients and Tangents to Level Curves



Gradients and Tangents to Level Curves



At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0).

Gradients and Tangents to Level Curves

This observation enables us to find equations for tangent lines to level curves. They are the lines normal to the gradients.

The line through a point $P_0(x_0, y_0)$ normal to a vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$ has the equation

$$A(x - x_0) + B(y - y_0) = 0.$$

Gradients and Tangents to Level Curves

This observation enables us to find equations for tangent lines to level curves. They are the lines normal to the gradients.

The line through a point $P_0(x_0, y_0)$ normal to a vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$ has the equation

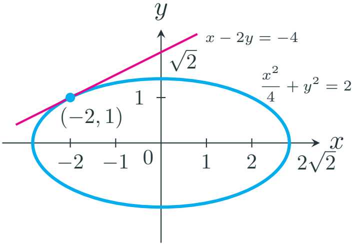
$$A(x - x_0) + B(y - y_0) = 0.$$

If $\mathbf{N}$ is the gradient $(\nabla f)\big|_{(x_0,y_0)} = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$, the equation is the tangent line at $P_0(x_0, y_0)$ given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0.$$

EXAMPLE

Find an equation for the tangent to the ellipse $\dfrac{x^2}{4} + y^2 = 2$ at the point $(-2, 1)$.



EXAMPLE

Find an equation for the tangent to the ellipse $\frac{x^2}{4} + y^2 = 2$ at the point $(-2, 1)$.

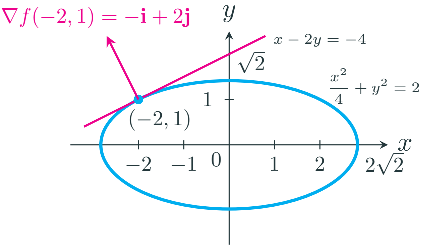
Solution:

The ellipse is a level curve of the function

$$f(x, y) = \frac{x^2}{4} + y^2.$$

The gradient of f at $(-2, 1)$ is

$$\nabla f\Big|_{(-2,1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j}\right)\Big|_{(-2,1)} = -\mathbf{i} + 2\mathbf{j}.$$



EXAMPLE

Find an equation for the tangent to the ellipse $\frac{x^2}{4} + y^2 = 2$ at the point $(-2, 1)$.

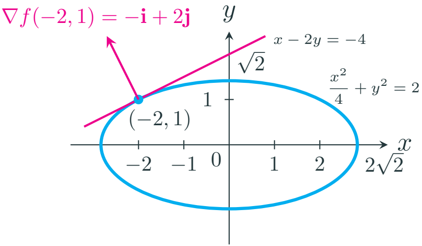
Solution:

The ellipse is a level curve of the function

$$f(x, y) = \frac{x^2}{4} + y^2.$$

The gradient of f at $(-2, 1)$ is

$$\nabla f\big|_{(-2,1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j}\right)\Big|_{(-2,1)} = -\mathbf{i}+2\mathbf{j}.$$



The tangent is the line

$$(-1)(x + 2) + (2)(y - 1) = 0$$

$$x - 2y = -4.$$

Tangent Planes and Normal Lines

Tangent Planes and Normal Lines

If $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f, then $f(g(t), h(t), k(t)) = c$. Differentiating both sides of this equation with respect to t leads to

$$\frac{d}{dt} f(g(t), h(t), k(t)) = \frac{d}{dt}(c) \xrightarrow{\text{Chain Rule}} \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j} + \frac{dk}{dt}\mathbf{k}\right)}_{d\mathbf{r}/dt} = 0.$$

Tangent Planes and Normal Lines

If $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f, then $f(g(t), h(t), k(t)) = c$. Differentiating both sides of this equation with respect to t leads to

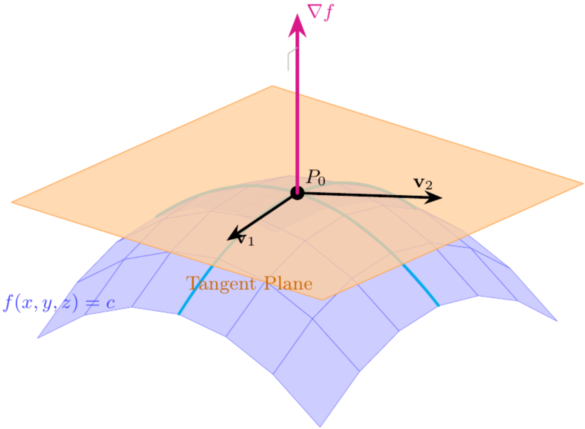
$$\frac{d}{dt} f(g(t), h(t), k(t)) = \frac{d}{dt}(c) \xrightarrow{\text{Chain Rule}} \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j} + \frac{dk}{dt}\mathbf{k}\right)}_{d\mathbf{r}/dt} = 0.$$

At every point along the curve, ∇f is orthogonal to the curve's velocity vector.

Tangent Planes and Normal Lines

The velocity vectors at P_0 all lie in a common plane



Tangent Planes and Normal Lines

Tangent Plane and Normal Line

The tangent plane at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f\big|_{P_0}$.
The normal line of the surface at P_0 is the line through P_0 parallel to $\nabla f\big|_{P_0}$.

Tangent Planes and Normal Lines

Tangent Plane and Normal Line

The tangent plane at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f\big|_{P_0}$.

The normal line of the surface at P_0 is the line through P_0 parallel to $\nabla f\big|_{P_0}$.

Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0.$$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t.$$

EXAMPLE

Find the tangent plane and normal line of the surface (A circular paraboloid)

$$f(x,y,z) = x^2 + y^2 + z - 9 = 0$$

at the point $P_0(1,2,4)$.

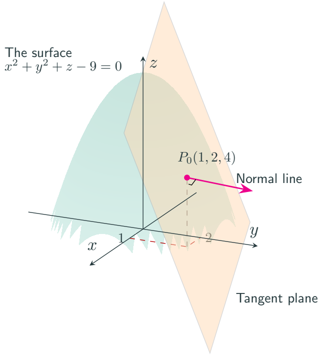
EXAMPLE

Find the tangent plane and normal line of the surface (A circular paraboloid)

$$f(x,y,z) = x^2 + y^2 + z - 9 = 0$$

at the point $P_0(1,2,4)$.

Solution: $\nabla f\big|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\big|_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$



EXAMPLE

Find the tangent plane and normal line of the surface (A circular paraboloid)

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0$$

at the point $P_0(1, 2, 4)$.

Solution: The tangent plane is therefore the plane

$$2(x-1) + 4(y-2) + (z-4) = 0, \quad \text{or} \quad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \quad y = 2 + 4t, \quad z = 4 + t.$$

Tangent Planes and Normal Lines

Tangent plane to a smooth surface $z = f(x, y)$ at a point $P_0(x_0, y_0, f(x_0, y_0))$ is the tangent plane through P_0 to the zero level surface of the function $F(x, y, z) = f(x, y) - z$.

Tangent Planes and Normal Lines

Tangent plane to a smooth surface $z = f(x, y)$ at a point $P_0(x_0, y_0, f(x_0, y_0))$ is the tangent plane through P_0 to the zero level surface of the function $F(x, y, z) = f(x, y) - z$.

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x}, \quad \frac{\partial F}{\partial y} = \frac{\partial f}{\partial y}, \quad \frac{\partial F}{\partial z} = -1 \Longrightarrow \nabla F = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} - \mathbf{k}$$

So, the plane has equation

$$\frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) - (z - z_0) = 0.$$

EXAMPLE

Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0, 0, 0)$.

EXAMPLE

Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0,0,0)$.

Solution: Let $f(x,y) = x\cos y - ye^x$

$$f_x(0,0) = (\cos y - ye^x)\Big|_{(0,0)} = 1 - 0 \cdot 1 = 1$$

$$f_y(0,0) = (-x\sin y - e^x)\Big|_{(0,0)} = 0 - 1 = -1.$$

EXAMPLE

Find the plane tangent to the surface $z = x \cos y - ye^x$ at $(0, 0, 0)$.

Solution: Let $f(x, y) = x \cos y - ye^x$

$$f_x(0,0) = (\cos y - ye^x)\Big|_{(0,0)} = 1 - 0 \cdot 1 = 1$$
$$f_y(0,0) = (-x \sin y - e^x)\Big|_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0,$$

or

$$x - y - z = 0.$$

EXAMPLE

The surfaces

$f(x, y, z) = x^2 + y^2 - 2 = 0$ A cylinder

and

$g(x, y, z) = x + z - 4 = 0$ A plane

meet in an ellipse E. Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

Solution:

EXAMPLE

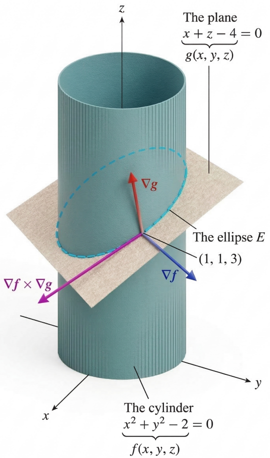
The surfaces

$f(x, y, z) = x^2 + y^2 - 2 = 0$ A cylinder

and

$g(x, y, z) = x + z - 4 = 0$ A plane

meet in an ellipse E. Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.



EXAMPLE

The surfaces

$$f(x,y,z) = x^2 + y^2 - 2 = 0 \qquad \text{and} \qquad g(x,y,z) = x + z - 4 = 0$$

meet in an ellipse E. Find parametric equations for the line tangent to E at the point $P_0(1,1,3)$.

Solution:

$$\nabla f\big|_{(1,1,3)} = (2x\mathbf{i} + 2y\mathbf{j})\big|_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g\big|_{(1,1,3)} = (\mathbf{i} + \mathbf{k})\big|_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

EXAMPLE

The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0 \qquad \text{and} \qquad g(x, y, z) = x + z - 4 = 0$$

meet in an ellipse E. Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

Solution:

$$\nabla f\big|_{(1,1,3)} = (2x\mathbf{i} + 2y\mathbf{j})\big|_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g\big|_{(1,1,3)} = (\mathbf{i} + \mathbf{k})\big|_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

The tangent line is for $t \in \mathbb{R}$

$$x = 1 + 2t, \quad y = 1 - 2t, \quad z = 3 - 2t.$$

EXAMPLE

Let $f(x, y, z) = x^2 + y^2 + z^2$.

(a) Find $\nabla f(x, y, z)$ and $\nabla f(1, -1, 2)$.

(b) Find an equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 6$ at the point $(1, -1, 2)$.

(c) What is the maximum rate of increase of f at $(1, -1, 2)$?

(d) What is the rate of change with respect to distance of f at $(1, -1, 2)$ measured in the direction from that point toward the point $(3, 1, 1)$?

EXAMPLE

Let $f(x, y, z) = x^2 + y^2 + z^2$.

(a) Find $\nabla f(x, y, z)$ and $\nabla f(1, -1, 2)$.

(b) Find an equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 6$ at the point $(1, -1, 2)$.

(c) What is the maximum rate of increase of f at $(1, -1, 2)$?

(d) What is the rate of change with respect to distance of f at $(1, -1, 2)$ measured in the direction from that point toward the point $(3, 1, 1)$?

Solution:

(a) $\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, so $\nabla f(1, -1, 2) = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

EXAMPLE

Let $f(x, y, z) = x^2 + y^2 + z^2$.

(a) Find $\nabla f(x, y, z)$ and $\nabla f(1, -1, 2)$.

(b) Find an equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 6$ at the point $(1, -1, 2)$.

(c) What is the maximum rate of increase of f at $(1, -1, 2)$?

(d) What is the rate of change with respect to distance of f at $(1, -1, 2)$ measured in the direction from that point toward the point $(3, 1, 1)$?

Solution:

(a) $\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, so $\nabla f(1, -1, 2) = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

(b) The required tangent plane has $\nabla f(1, -1, 2)$ as normal. Therefore, its equation is given by

$$2(x - 1) - 2(y + 1) + 4(z - 2) = 0$$

or, more simply, $x - y + 2z = 6$.

EXAMPLE

Let $f(x, y, z) = x^2 + y^2 + z^2$.

(a) Find $\nabla f(x, y, z)$ and $\nabla f(1, -1, 2)$.

(b) Find an equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 6$ at the point $(1, -1, 2)$.

(c) What is the maximum rate of increase of f at $(1, -1, 2)$?

(d) What is the rate of change with respect to distance of f at $(1, -1, 2)$ measured in the direction from that point toward the point $(3, 1, 1)$?

Solution:

(c) The maximum rate of increase of f at $(1, -1, 2)$ is $|\nabla f(1, -1, 2)| = 2\sqrt{6}$, and it occurs in the direction of the vector $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

EXAMPLE

Let $f(x, y, z) = x^2 + y^2 + z^2$.

(a) Find $\nabla f(x, y, z)$ and $\nabla f(1, -1, 2)$.

(b) Find an equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 6$ at the point $(1, -1, 2)$.

(c) What is the maximum rate of increase of f at $(1, -1, 2)$?

(d) What is the rate of change with respect to distance of f at $(1, -1, 2)$ measured in the direction from that point toward the point $(3, 1, 1)$?

Solution:

(d) The direction from $(1, -1, 2)$ toward $(3, 1, 1)$ is specified by $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The rate of change of f with respect to distance in this direction is

$$\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{4 + 4 + 1}} \cdot (2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = \frac{4 - 4 - 4}{3} = -\frac{4}{3},$$

that is, f decreases at rate $4/3$ of a unit per horizontal unit moved.

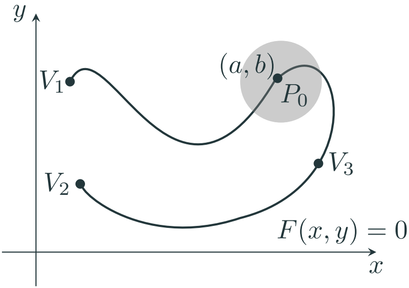
Implicit Functions

Implicit Functions

Let $F(x, y)$ be a differentiable function of two variables x and y.

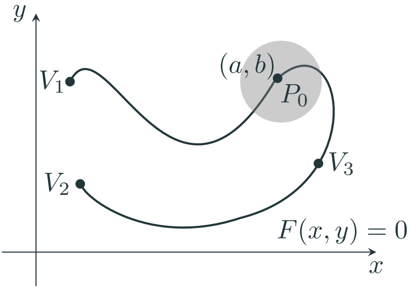
Implicit Functions

Let $F(x, y)$ be a differentiable function of two variables x and y. Let $F(a, b) = 0$ and suppose the equation $F(x, y) = 0$ can be solved for y as a function of x.



Implicit Functions

Let $F(x, y)$ be a differentiable function of two variables x and y. Let $F(a, b) = 0$ and suppose the equation $F(x, y) = 0$ can be solved for y as a function of x.

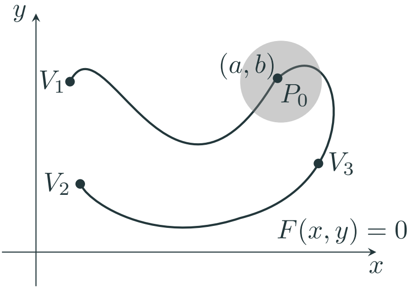


Differentiating w.r.t. x gives

$$F_1(x, y) + F_2(x, y)\frac{dy}{dx} = 0.$$

Implicit Functions

Let $F(x, y)$ be a differentiable function of two variables x and y. Let $F(a, b) = 0$ and suppose the equation $F(x, y) = 0$ can be solved for y as a function of x.



Differentiating w.r.t. x gives

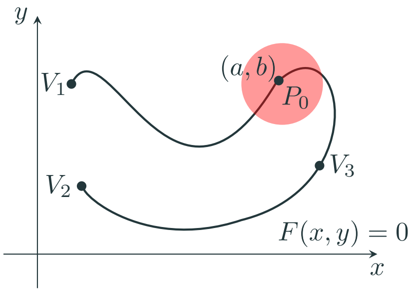
$$F_1(x, y) + F_2(x, y)\frac{dy}{dx} = 0.$$

Thus

$$\left.\frac{dy}{dx}\right|_{x=a} = -\frac{F_1(a, b)}{F_2(a, b)}$$

if $F_2(a, b) \neq 0$.

Implicit Functions



Differentiating w.r.t. x gives

$$F_1(x,y) + F_2(x,y)\frac{dy}{dx} = 0.$$

Thus

$$\left.\frac{dy}{dx}\right|_{x=a} = -\frac{F_1(a,b)}{F_2(a,b)}$$

if $F_2(a,b) \neq 0$.

$\boxed{F_2(a,b) \neq 0}$ $\longrightarrow$ enough to guarantee $F(x,y)=0$ can besolved for y as a function of x near P_0.

Implicit Functions

Suppose the equation $F(x, y, z) = 0$ defines z as a function of x and y near a point $P_0 = (x_0, y_0, z_0)$ with coordinates satisfying the equation.

Implicit Functions

Suppose the equation $F(x, y, z) = 0$ defines z as a function of x and y near a point $P_0 = (x_0, y_0, z_0)$ with coordinates satisfying the equation. Differentiating the both sides of the equation $F(x, y, z) = 0$ w.r.t. x and y, we get

$$F_1(x, y, z) + F_3(x, y, z)\frac{\partial z}{\partial x} = 0$$
$$F_2(x, y, z) + F_3(x, y, z)\frac{\partial z}{\partial y} = 0.$$

Implicit Functions

Suppose the equation $F(x, y, z) = 0$ defines z as a function of x and y near a point $P_0 = (x_0, y_0, z_0)$ with coordinates satisfying the equation. Differentiating the both sides of the equation $F(x, y, z) = 0$ w.r.t. x and y, we get

$$F_1(x, y, z) + F_3(x, y, z)\frac{\partial z}{\partial x} = 0$$
$$F_2(x, y, z) + F_3(x, y, z)\frac{\partial z}{\partial y} = 0.$$

Thus

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)} = -\frac{F_1(x_0, y_0, z_0)}{F_3(x_0, y_0, z_0)} \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(x_0, y_0)} = -\frac{F_2(x_0, y_0, z_0)}{F_3(x_0, y_0, z_0)}$$

if $F_3(x_0, y_0, z_0) \neq 0$.

EXMAPLE

Near what points on the sphere $x^2 + y^2 + z^2 = 1$ can the equation of the sphere be solved for z as a function of x and y? Find $\partial z/\partial x$ and $\partial z/\partial y$ at such points.

EXMAPLE

Near what points on the sphere $x^2 + y^2 + z^2 = 1$ can the equation of the sphere be solved for z as a function of x and y? Find $\partial z/\partial x$ and $\partial z/\partial y$ at such points.

Solution: The sphere is the level surface $F(x, y, z) = 0$, where $F(x, y, z) = x^2 + y^2 + z^2 - 1$.

EXMAPLE

Near what points on the sphere $x^2 + y^2 + z^2 = 1$ can the equation of the sphere be solved for z as a function of x and y? Find $\partial z/\partial x$ and $\partial z/\partial y$ at such points.

Solution: The sphere is the level surface $F(x, y, z) = 0$, where $F(x, y, z) = x^2 + y^2 + z^2 - 1$. Since

$$F_3(x, y, z) = 2z$$

then

$$F_3(x, y, z) = 0 \iff z = 0$$

Thus the equation $F(x, y, z) = 0$ can be solved for $z = z(x, y)$ near P_0 if P_0 does not lie on the circle $x^2 + y^2 = 1, z = 0$, that is the equator of the sphere.

Implicit Functions

EXMAPLE

Near what points on the sphere $x^2 + y^2 + z^2 = 1$ can the equation of the sphere be solved for z as a function of x and y? Find $\partial z/\partial x$ and $\partial z/\partial y$ at such points.

Solution: If $z \neq 0$, we can calculate the partial derivatives of the solution $z = z(x, y)$ by implicitly differentiating the equation of the sphere: $x^2 + y^2 + z^2 = 1$:

$$2x + 2z\frac{\partial z}{\partial x} = 0, \qquad \text{so} \qquad \frac{\partial z}{\partial x} = -\frac{x}{z},$$

$$2y + 2z\frac{\partial z}{\partial y} = 0, \qquad \text{so} \qquad \frac{\partial z}{\partial y} = -\frac{y}{z}.$$

Implicit Functions

When we are given $F(x, y, z) = 0$ and asked $\dfrac{\partial x}{\partial z}$, we could understand that x is intended to be a function of y and z.

Implicit Functions

When we are given $F(x, y, z) = 0$ and asked $\dfrac{\partial x}{\partial z}$, we could understand that x is intended to be a function of y and z.

What if we are given a system

$$\begin{cases} F(x, y, z, w) = 0 \\ G(x, y, z, w) = 0 \end{cases}$$

and asked $\dfrac{\partial x}{\partial z}$?

Implicit Functions

When we are given $F(x, y, z) = 0$ and asked $\dfrac{\partial x}{\partial z}$, we could understand that x is intended to be a function of y and z.

What if we are given a system

$$\begin{cases} F(x, y, z, w) = 0 \\ G(x, y, z, w) = 0 \end{cases}$$

and asked $\dfrac{\partial x}{\partial z}$? In short, which of the situations

$$\begin{cases} x = x(z, w) \\ y = y(z, w) \end{cases} \quad \text{or} \quad \begin{cases} x = x(y, z) \\ w = w(y, z) \end{cases}$$

are we dealing with?

Implicit Functions

We use subscripts to specify which variable is to be regarded as the other independent variable.

$$\left(\frac{\partial x}{\partial z}\right)_w \text{ implies the interpretation } \begin{cases} x = x(z, w) \\ y = y(z, w), \end{cases}$$

$$\left(\frac{\partial x}{\partial z}\right)_y \text{ implies the interpretation } \begin{cases} x = x(y, z) \\ w = w(y, z). \end{cases}$$

Implicit Functions Example

EXAMPLE

Given the equations $F(x, y, z, w) = 0$ and $G(x, y, z, w) = 0$, where F and G have continuous first partial derivatives, calculate $\left(\dfrac{\partial x}{\partial z}\right)_w$.

Implicit Functions Example

EXAMPLE

Given the equations $F(x, y, z, w) = 0$ and $G(x, y, z, w) = 0$, where F and G have continuous first partial derivatives, calculate $\left(\dfrac{\partial x}{\partial z}\right)_w$.

Solution: We differentiate the two equations with respect to z, regarding x and y as functions of z and w, and holding w fixed:

$$
\begin{aligned}
F_1 \frac{\partial x}{\partial z} + F_2 \frac{\partial y}{\partial z} + F_3 = 0 \\
G_1 \frac{\partial x}{\partial z} + G_2 \frac{\partial y}{\partial z} + G_3 = 0
\end{aligned}
$$

Implicit Functions Example

EXAMPLE

Given the equations $F(x, y, z, w) = 0$ and $G(x, y, z, w) = 0$, where F and G have continuous first partial derivatives, calculate $\left(\frac{\partial x}{\partial z}\right)_w$.

Solution: We differentiate the two equations with respect to z, regarding x and y as functions of z and w, and holding w fixed:

$$F_1 \frac{\partial x}{\partial z} + F_2 \frac{\partial y}{\partial z} + F_3 = 0$$
$$G_1 \frac{\partial x}{\partial z} + G_2 \frac{\partial y}{\partial z} + G_3 = 0$$

Eliminating $\frac{\partial y}{\partial z}$, we obtain

$$\left(\frac{\partial x}{\partial z}\right)_w = -\frac{F_3 G_2 - F_2 G_3}{F_1 G_2 - F_2 G_1}.$$

Implicit Functions Example

EXAMPLE

Let x, y, u, and v be related by the equations

$$\begin{cases} u = x^2 + xy - y^2 \\ v = 2xy + y^2. \end{cases}$$

Find (a) $\left(\frac{\partial x}{\partial u}\right)_v$ and (b) $\left(\frac{\partial x}{\partial u}\right)_y$ at the point where $x = 2$ and $y = -1$.

Implicit Functions Example

EXAMPLE

Let x, y, u, and v be related by the equations

$$\begin{cases} u = x^2 + xy - y^2 \\ v = 2xy + y^2. \end{cases}$$

Find (a) $\left(\frac{\partial x}{\partial u}\right)_v$ and (b) $\left(\frac{\partial x}{\partial u}\right)_y$ at the point where $x = 2$ and $y = -1$.

Solution: (a) To calculate $(\partial x/\partial u)_v$ we regard x and y as functions of u and v and differentiate the given equations with respect to u, holding v constant:

$$1 = \frac{\partial u}{\partial u} = (2x + y)\frac{\partial x}{\partial u} + (x - 2y)\frac{\partial y}{\partial u}$$
$$0 = \frac{\partial v}{\partial u} = 2y\frac{\partial x}{\partial u} + (2x + 2y)\frac{\partial y}{\partial u}$$

Implicit Functions Example

EXAMPLE

Let x, y, u, and v be related by the equations

$$\begin{cases} u = x^2 + xy - y^2 \\ v = 2xy + y^2. \end{cases}$$

Find (a) $\left(\dfrac{\partial x}{\partial u}\right)_v$ and (b) $\left(\dfrac{\partial x}{\partial u}\right)_y$ at the point where $x = 2$ and $y = -1$.

Solution: (a) To calculate $(\partial x/\partial u)_v$ we regard x and y as functions of u and v and differentiate the given equations with respect to u, holding v constant:

$$1 = \frac{\partial u}{\partial u} = (2x + y)\frac{\partial x}{\partial u} + (x - 2y)\frac{\partial y}{\partial u}$$
$$0 = \frac{\partial v}{\partial u} = 2y\frac{\partial x}{\partial u} + (2x + 2y)\frac{\partial y}{\partial u}$$

$$\longrightarrow$$

At $x = 2, y = -1$ we have

$$1 = 3\frac{\partial x}{\partial u} + 4\frac{\partial y}{\partial u}$$
$$0 = -2\frac{\partial x}{\partial u} + 2\frac{\partial y}{\partial u}.$$

Implicit Functions Example

EXAMPLE

Let x, y, u, and v be related by the equations

$$\begin{cases} u = x^2 + xy - y^2 \\ v = 2xy + y^2. \end{cases}$$

Find (a) $\left(\dfrac{\partial x}{\partial u}\right)_v$ and (b) $\left(\dfrac{\partial x}{\partial u}\right)_y$ at the point where $x = 2$ and $y = -1$.

Solution: (a) To calculate $(\partial x / \partial u)_v$ we regard x and y as functions of u and v and differentiate the given equations with respect to u, holding v constant:

$$1 = \frac{\partial u}{\partial u} = (2x + y)\frac{\partial x}{\partial u} + (x - 2y)\frac{\partial y}{\partial u}$$
$$0 = \frac{\partial v}{\partial u} = 2y\frac{\partial x}{\partial u} + (2x + 2y)\frac{\partial y}{\partial u}$$

$\longrightarrow$ At $x = 2, y = -1$ we have

$$1 = 3\frac{\partial x}{\partial u} + 4\frac{\partial y}{\partial u}$$
$$0 = -2\frac{\partial x}{\partial u} + 2\frac{\partial y}{\partial u}.$$

Eliminating $\partial y / \partial u$ leads to the result $(\partial x / \partial u)_v = 1/7$.

Implicit Functions Example

EXAMPLE

Let x, y, u, and v be related by the equations

$$\begin{cases} u = x^2 + xy - y^2 \\ v = 2xy + y^2. \end{cases}$$

Find (a) $\left(\frac{\partial x}{\partial u}\right)_v$ and (b) $\left(\frac{\partial x}{\partial u}\right)_y$ at the point where $x = 2$ and $y = -1$.

Solution: (b) To calculate $\left(\frac{\partial x}{\partial u}\right)_y$ we regard x and v as functions of y and u and differentiate the given equations with respect to u, holding y constant:

$$1 = (2x + y)\frac{\partial x}{\partial u}, \quad \frac{\partial v}{\partial u} = 2y\frac{\partial x}{\partial u}.$$

Implicit Functions Example

EXAMPLE

Let x, y, u, and v be related by the equations

$$\begin{cases} u = x^2 + xy - y^2 \\ v = 2xy + y^2. \end{cases}$$

Find (a) $\left(\frac{\partial x}{\partial u}\right)_v$ and (b) $\left(\frac{\partial x}{\partial u}\right)_y$ at the point where $x = 2$ and $y = -1$.

Solution: (b) To calculate $\left(\frac{\partial x}{\partial u}\right)_y$ we regard x and v as functions of y and u and differentiate the given equations with respect to u, holding y constant:

$$1 = (2x + y)\frac{\partial x}{\partial u}, \quad \frac{\partial v}{\partial u} = 2y\frac{\partial x}{\partial u}.$$

At $x = 2$, $y = -1$ the first equation immediately gives $\left(\frac{\partial x}{\partial u}\right)_y = 1/3$.