MAT124 MATHEMATICS II

Implicit Functions, Applications of Partial Derivatives - Extreme Values

Outline

Implicit Functions

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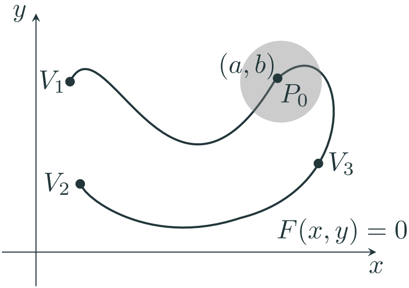

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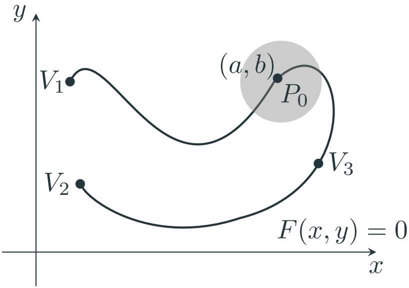

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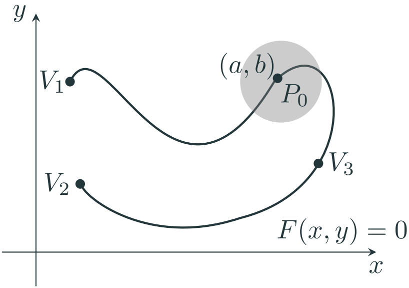

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Thus

$$\left.\frac{dy}{dx}\right|_{x=a} = -\frac{F_1(a, b)}{F_2(a, b)}$$

if $F_2(a, b) \neq 0$.

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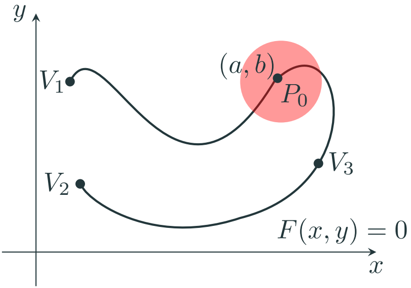

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$\boxed{F_2(a, b) \neq 0}$ $\longrightarrow$ enough to guarantee $F(x, y) = 0$ can besolved for y as a function of x near P_0.

Implicit Functions

Suppose the equation $F(x, y, z) = 0$ defines z as a function of x and y near a point $P_0 = (x_0, y_0, z_0)$ with coordinates satisfying the equation.

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$$F_1(x, y, z) + F_3(x, y, z)\frac{\partial z}{\partial x} = 0$$
$$F_2(x, y, z) + F_3(x, y, z)\frac{\partial z}{\partial y} = 0.$$

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$$\begin{aligned} F_1(x, y, z) + F_3(x, y, z)\frac{\partial z}{\partial x} &= 0 \\ F_2(x, y, z) + F_3(x, y, z)\frac{\partial z}{\partial y} &= 0. \end{aligned}$$

Thus

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)} = -\frac{F_1(x_0, y_0, z_0)}{F_3(x_0, y_0, z_0)} \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(x_0, y_0)} = -\frac{F_2(x_0, y_0, z_0)}{F_3(x_0, y_0, z_0)}$$

if $F_3(x_0, y_0, z_0) \neq 0$.

Implicit Functions

EXMAPLE

Near what points on the sphere $x^2 + y^2 + z^2 = 1$ can the equation of the sphere be solved for z as a function of x and y? Find $\partial z/\partial x$ and $\partial z/\partial y$ at such points.

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Solution: The sphere is the level surface $F(x, y, z) = 0$, where $F(x, y, z) = x^2 + y^2 + z^2 - 1$.

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Solution: The sphere is the level surface $F(x, y, z) = 0$, where $F(x, y, z) = x^2 + y^2 + z^2 - 1$. Since

$$F_3(x, y, z) = 2z$$

then

$$F_3(x, y, z) = 0 \iff z = 0$$

Thus the equation $F(x, y, z) = 0$ can be solved for $z = z(x, y)$ near P_0 if P_0 does not lie on the circle $x^2 + y^2 = 1, z = 0$, that is the equator of the sphere.

EXMAPLE

Near what points on the sphere $x^2 + y^2 + z^2 = 1$ can the equation of the sphere be solved for z as a function of x and y? Find $\partial z/\partial x$ and $\partial z/\partial y$ at such points.

Solution: If $z \neq 0$, we can calculate the partial derivatives of the solution $z = z(x, y)$ by implicitly differentiating the equation of the sphere: $x^2 + y^2 + z^2 = 1$:

$$2x + 2z\frac{\partial z}{\partial x} = 0, \qquad \text{so} \qquad \frac{\partial z}{\partial x} = -\frac{x}{z},$$

$$2y + 2z\frac{\partial z}{\partial y} = 0, \qquad \text{so} \qquad \frac{\partial z}{\partial y} = -\frac{y}{z}.$$

Implicit Functions

When we are given $F(x, y, z) = 0$ and asked $\dfrac{\partial x}{\partial z}$, we could understand that x is intended to be a function of y and z.

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What if we are given a system

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and asked $\dfrac{\partial x}{\partial z}$? In short, which of the situations

$$\begin{cases} x = x(z, w) \\ y = y(z, w) \end{cases} \quad \text{or} \quad \begin{cases} x = x(y, z) \\ w = w(y, z) \end{cases}$$

are we dealing with?

Implicit Functions

We use subscripts to specify which variable is to be regarded as the other independent variable.

$$\left(\frac{\partial x}{\partial z}\right)_w \text{ implies the interpretation } \begin{cases} x = x(z, w) \\ y = y(z, w), \end{cases}$$

$$\left(\frac{\partial x}{\partial z}\right)_y \text{ implies the interpretation } \begin{cases} x = x(y, z) \\ w = w(y, z). \end{cases}$$

Implicit Functions Example

EXAMPLE

Given the equations $F(x, y, z, w) = 0$ and $G(x, y, z, w) = 0$, where F and G have continuous first partial derivatives, calculate $\left(\dfrac{\partial x}{\partial z}\right)_w$.

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Solution: We differentiate the two equations with respect to z, regarding x and y as functions of z and w, and holding w fixed:

$$F_1 \frac{\partial x}{\partial z} + F_2 \frac{\partial y}{\partial z} + F_3 = 0$$
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Eliminating $\dfrac{\partial y}{\partial z}$, we obtain

$$\left(\frac{\partial x}{\partial z}\right)_w = -\frac{F_3 G_2 - F_2 G_3}{F_1 G_2 - F_2 G_1}.$$

Implicit Functions Example

EXAMPLE

Let x, y, u, and v be related by the equations

$$\begin{cases} u = x^2 + xy - y^2 \\ v = 2xy + y^2. \end{cases}$$

Find (a) $\left(\dfrac{\partial x}{\partial u}\right)_v$ and (b) $\left(\dfrac{\partial x}{\partial u}\right)_y$ at the point where $x = 2$ and $y = -1$.

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Solution: (a) To calculate $(\partial x/\partial u)_v$ we regard x and y as functions of u and v and differentiate the given equations with respect to u, holding v constant:

$$1 = \frac{\partial u}{\partial u} = (2x + y)\frac{\partial x}{\partial u} + (x - 2y)\frac{\partial y}{\partial u}$$

$$0 = \frac{\partial v}{\partial u} = 2y\frac{\partial x}{\partial u} + (2x + 2y)\frac{\partial y}{\partial u}$$

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$\longrightarrow$ At $x = 2, y = -1$ we have

$$1 = 3\frac{\partial x}{\partial u} + 4\frac{\partial y}{\partial u}$$
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Eliminating $\partial y/\partial u$ leads to the result $(\partial x/\partial u)_v = 1/7$.

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$$1 = (2x + y)\frac{\partial x}{\partial u}, \quad \frac{\partial v}{\partial u} = 2y\frac{\partial x}{\partial u}.$$

At $x = 2$, $y = -1$ the first equation immediately gives $\left(\frac{\partial x}{\partial u}\right)_y = 1/3$.

Jacobian determinants

The Jacobian determinant (or simply the Jacobian) of the two functions, $u = u(x, y)$ and $v = v(x, y)$, with respect to two variables, x and y, is the determinant

$$\frac{\partial(u, v)}{\partial(x, y)} := \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}.$$

Similarly, the Jacobian of two functions, $F(x, y, \dots)$ and $G(x, y, \dots)$, with respect to the variables, x and y, is the determinant

$$\frac{\partial(F, G)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{vmatrix} = \begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix}.$$

Jacobian determinants

The definition above can be extended in the obvious way to give the Jacobian of n functions with respect to n variables. For example, the Jacobian of three functions, $F(x, y, z, \dots)$, $G(x, y, z, \dots)$, and $H(x, y, z, \dots)$, with respect to variables, x, y, and z, is the determinant

$$\frac{\partial(F, G, H)}{\partial(x, y, z)} = \begin{vmatrix} F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \\ H_1 & H_2 & H_3 \end{vmatrix}.$$

EXAMPLE

Consider the system of linear equations

$$\begin{aligned} F(x,y,s,t) &= a_1x + b_1y + c_1s + d_1t + e_1 = 0 \\ G(x,y,s,t) &= a_2x + b_2y + c_2s + d_2t + e_2 = 0. \end{aligned}$$

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$$G(x, y, s, t) = a_2 x + b_2 y + c_2 s + d_2 t + e_2 = 0.$$

This system can be written in matrix form:

$$\mathcal{A}\begin{bmatrix} x \\ y \end{bmatrix} + \mathcal{C}\begin{bmatrix} s \\ t \end{bmatrix} + \mathcal{E} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where

$$\mathcal{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}, \quad \text{and} \quad \mathcal{E} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Implicit Function Theorem

EXAMPLE

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$$\det(\mathcal{A}) \neq 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = -\mathcal{A}^{-1}\left(\mathcal{C}\begin{bmatrix} s \\ t \end{bmatrix} + \mathcal{E}\right)$$

Observe that $\det(\mathcal{A}) = \partial(F, G)/\partial(x, y)$, so the nonvanishing of this Jacobian (nonzero) guarantees that the equations can be solved for x and y.

Implicit Function Theorem

Consider a system of n equations in $n + m$ variables,

$$\begin{cases} F_{(1)}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = 0 \\ F_{(2)}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = 0 \\ \vdots \\ F_{(n)}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = 0, \end{cases}$$

and a point $P_0 = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)$ that satisfies the system.

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and a point $P_0 = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)$ that satisfies the system. Suppose each of the functions $F_{(i)}$ has continuous first partial derivatives with respect to each of the variables x_j and y_k, $(i = 1, \ldots, n,\ j = 1, \ldots, m,\ k = 1, \ldots, n)$, near P_0. Finally, suppose that

$$\left.\frac{\partial(F_{(1)}, F_{(2)}, \ldots, F_{(n)})}{\partial(y_1, y_2, \ldots, y_n)}\right|_{P_0} \neq 0.$$

Then the system can be solved for $y_1, y_2, \ldots, y_n$ as functions of $x_1, x_2, \ldots, x_m$ near P_0.

Implicit Function Theorem

That is, there exist functions

$$\varphi_1(x_1, \ldots, x_m), \ldots, \varphi_n(x_1, \ldots, x_m)$$

such that

$$\varphi_j(a_1, \ldots, a_m) = b_j, \qquad (j = 1, \ldots, n),$$

and such that the equations

$$\begin{aligned} F_{(1)}\big(x_1, \ldots, x_m, \varphi_1(x_1, \ldots, x_m), \ldots, \varphi_n(x_1, \ldots, x_m)\big) &= 0, \\ F_{(2)}\big(x_1, \ldots, x_m, \varphi_1(x_1, \ldots, x_m), \ldots, \varphi_n(x_1, \ldots, x_m)\big) &= 0, \\ &\vdots \\ F_{(n)}\big(x_1, \ldots, x_m, \varphi_1(x_1, \ldots, x_m), \ldots, \varphi_n(x_1, \ldots, x_m)\big) &= 0, \end{aligned}$$

hold for all $(x_1, \ldots, x_m)$ sufficiently near $(a_1, \ldots, a_m)$.

Implicit Function Theorem

Moreover, for $i = 1, \dots, n$ and $j = 1, \dots, m$

$$\frac{\partial \varphi_i}{\partial x_j} = \left(\frac{\partial y_i}{\partial x_j}\right)_{x_1,\dots,x_{j-1},x_{j+1},\dots,x_m} = -\frac{\dfrac{\partial(F_{(1)}, F_{(2)}, \dots, F_{(n)})}{\partial(y_1, \dots, x_j, \dots, y_n)}}{\dfrac{\partial(F_{(1)}, F_{(2)}, \dots, F_{(n)})}{\partial(y_1, \dots, y_i, \dots, y_n)}}.$$

Implicit Function Theorem Example

EXAMPLE

Show that the system

$$\begin{cases} xy^2 + xzu + yv^2 = 3 \\ x^3yz + 2xv - u^2v^2 = 2 \end{cases}$$

can be solved for (u, v) as a (vector) function of (x, y, z) near the point P_0 where $(x, y, z, u, v) = (1, 1, 1, 1, 1)$, and find the value of $\frac{\partial v}{\partial y}$ for the solution at $(x, y, z) = (1, 1, 1)$.

Implicit Function Theorem Example

Solution: Let $P_0 = (1, 1, 1, 1, 1)$ and

$$\begin{cases} F(x, y, z, u, v) = xy^2 + xzu + yv^2 - 3 \\ G(x, y, z, u, v) = x^3yz + 2xv - u^2v^2 - 2 \end{cases}$$

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$$\left.\frac{\partial(F,G)}{\partial(u,v)}\right|_{P_0} = \left.\begin{vmatrix} xz & 2yv \\ -2uv^2 & 2x - 2u^2v \end{vmatrix}\right|_{P_0} = \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 4$$

$$\left.\frac{\partial(F,G)}{\partial(u,y)}\right|_{P_0} = \left.\begin{vmatrix} xz & 2xy + v^2 \\ -2uv^2 & x^3z \end{vmatrix}\right|_{P_0} = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 7$$

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$$\left(\frac{\partial v}{\partial y}\right)_{x,z} = -\frac{\left.\frac{\partial(F,G)}{\partial(u,y)}\right|_{P_0}}{\left.\frac{\partial(F,G)}{\partial(u,v)}\right|_{P_0}} = -\frac{7}{4}$$

Implicit Function Theorem Example

EXAMPLE

If the equations $x = u^2 + v^2$ and $y = uv$ are solved for u and v in terms of x and y, find, where possible,

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \text{and} \quad \frac{\partial v}{\partial y}.$$

Hence, show that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}},$$

provided the denominator is not zero.

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Hence, show that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}},$$

provided the denominator is not zero.

Solution: Since

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 2v \\ v & u \end{vmatrix} = 2u^2 - 2v^2$$

when $u^2 \neq v^2$, then $J \neq 0$ and we can calculate the required partial derivatives as follows

Implicit Function Theorem Example

Solution: $$\frac{\partial u}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)} = -\frac{1}{J}\begin{vmatrix} -1 & 2v \\ 0 & u \end{vmatrix} = \frac{u}{2(u^2 - v^2)}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)} = -\frac{1}{J}\begin{vmatrix} 0 & 2v \\ -1 & u \end{vmatrix} = \frac{-2v}{2(u^2 - v^2)}$$

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Implicit Function Theorem Example

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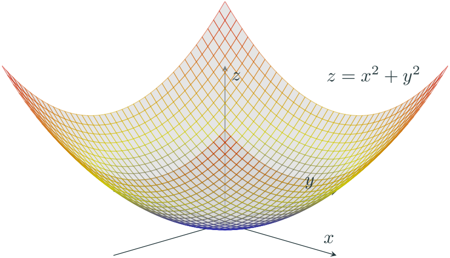
Finally,

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{J^2}\begin{vmatrix} u & -2v \\ -v & 2u \end{vmatrix} = \frac{J}{J^2} = \frac{1}{J} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$$

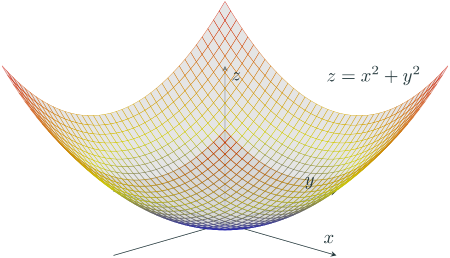
Applications of Partial Derivatives

Extreme Values



$x^2 + y^2$ has minimum value 0 at the origin

Extreme Values



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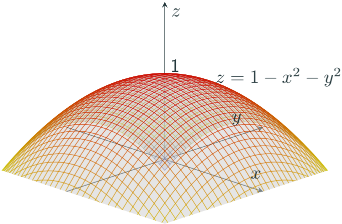

$1 - x^2 - y^2$ has maximum value 1 at the origin

Extreme Values

Extreme (Local - Absolute) Values

We say that a function of two variables has a local maximum value at the point (a, b) in its domain if $f(x, y) \leq f(a, b)$ for all points (x, y) in the domain of f that are sufficiently close to the point (a, b). If the inequality holds for all (x, y) in the domain of f, then we say that f has an absolute maximum value at (a, b).

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Similar definitions obtain for local and absolute minimum values. In practice, the word absolute is usually omitted, and we refer simply to the maximum or the minimum value of f.

Extreme Values

Theorem (Necessary conditions for extreme values)

A function $f(x, y)$ can have a local or absolute extreme value at a point (a, b) in its domain only if (a, b) is one of the following:

(a) a critical point of f, that is, a point satisfying $\nabla f(a, b) = 0$,

(b) a singular point of f, that is, a point where $\nabla f(a, b)$ does not exist, or

(c) a boundary point of the domain of f.

Extreme Values

A set in $\mathbb{R}^n$ is bounded if it is contained inside some *ball* $x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2$ of finite radius R. A set on the real line is bounded if it is contained in an interval of finite length.

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Theorem (Sufficient conditions for extreme values)

If f is a continuous function of n variables whose domain is a closed and bounded set in $\mathbb{R}^n$, then the range of f is a bounded set of real numbers, and there are points in its domain where f takes on absolute maximum and minimum values.

Extreme Values Examples

EXAMPLE

- The function $f(x, y) = x^2 + y^2$ has a critical point at $(0, 0)$, since $\nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j}$ and both components of ∇f vanish at $(0, 0)$. Since

$$f(x, y) > 0 = f(0, 0) \quad \text{if} \quad (x, y) \neq (0, 0),$$

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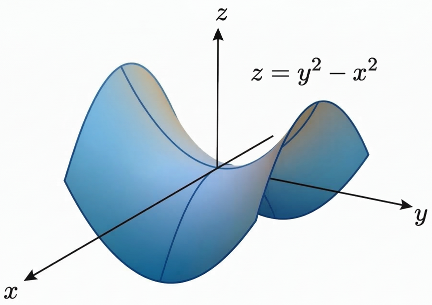
- Similarly, since $\nabla g = -2x\,\mathbf{i} - 2y\,\mathbf{j}$

$$g(x, y) = 1 - x^2 - y^2$$

has (absolute) maximum value 1 at its critical point $(0, 0)$.

EXAMPLE

The function $h(x, y) = y^2 - x^2$ also has a critical point at $(0, 0)$ but has neither a local maximum nor a local minimum value at that point. Observe that $h(0, 0) = 0$ but $h(x, 0) < 0$ and $h(0, y) > 0$ for all nonzero values of x and y. The graph of h is a hyperbolic paraboloid. In view of the shape of this surface, we call the critical point $(0, 0)$ a saddle point of h.



Extreme Values - Saddle point

In general, we will somewhat loosely call any interior critical point of the domain of a function f of several variables a saddle point if f does not have a local maximum or minimum value there.

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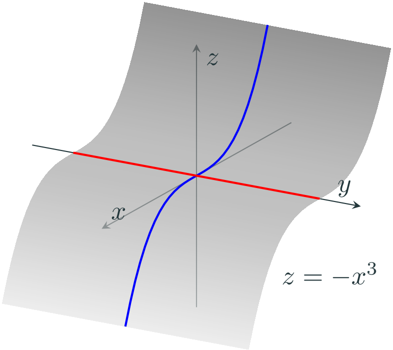
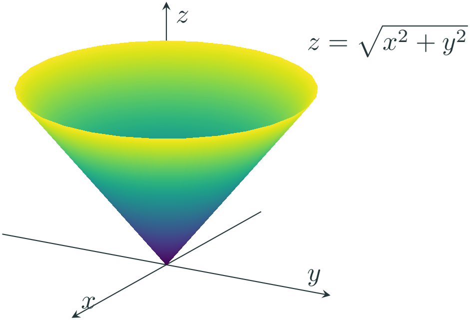
Near a saddle point, the graph will not always look like a saddle.



EXMAPLE

The function $f(x,y) = \sqrt{x^2+y^2}$ has no critical points but does have a singular point at $(0,0)$ where it has a local (and absolute) minimum value, zero. The graph of f is (one nappe of) a circular cone.



Extreme Values Examples

EXMAPLE

The function $f(x, y) = 1 - x$ is defined everywhere in the xy-plane and has no critical or singular points. ($\nabla f(x, y) = -\mathbf{i}$ at every point (x, y).) Therefore f has no extreme values.

Extreme Values Examples

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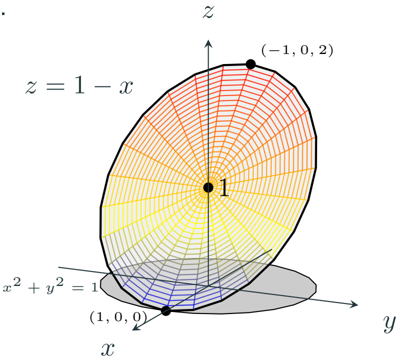
The function $f(x, y) = 1 - x$ is defined everywhere in the xy-plane and has no critical or singular points. ($\nabla f(x, y) = -\mathbf{i}$ at every point (x, y).) Therefore f has no extreme values.However, if we restrict the domain of f to the points in the disk $x^2 + y^2 \leq 1$ (a closed bounded set in the xy-plane), then f does have absolute maximum and minimum values. The maximum value is 2 at the boundary point $(-1, 0)$ and the minimum value is 0 at $(1, 0)$.



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$$\begin{aligned} 0 = f_1(x, y) = 6x^2 - 6y &\qquad \iff \qquad x^2 = y \\ 0 = f_2(x, y) = -6x + 6y &\qquad \iff \qquad x = y. \end{aligned}$$

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Together, these equations imply that $x^2 = x$ so that $x = 0$ or $x = 1$. Therefore, the critical points are $(0,0)$ and $(1,1)$.

Consider the point $(0,0)$. Let Δf be

$$\Delta f = f(h,k) - f(0,0) = 2h^3 - 6hk + 3k^2.$$

Since $f(h,0) - f(0,0) = 2h^3$ is positive for small positive h and negative for small negative h, f cannot have a maximum or minimum value at $(0,0)$. Therefore, $(0,0)$ is a saddle point.

Extreme Values Examples

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Find and classify the critical points of $f(x, y) = 2x^3 - 6xy + 3y^2$.

Solution: Now consider $(1, 1)$. Let Δf be

$$\begin{aligned}\Delta f &= f(1+h, 1+k) - f(1,1) \\ &= 2(1+h)^3 - 6(1+h)(1+k) + 3(1+k)^2 - (-1) \\ &= 2 + 6h + 6h^2 + 2h^3 - 6 - 6h - 6k - 6hk + 3 + 6k + 3k^2 + 1 \\ &= 6h^2 - 6hk + 3k^2 + 2h^3 \\ &= 3(h-k)^2 + h^2(3+2h).\end{aligned}$$

Both terms in the latter expression are nonnegative if $|h| < 3/2$, and they are not both zero unless $h = k = 0$.

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Both terms in the latter expression are nonnegative if $|h| < 3/2$, and they are not both zero unless $h = k = 0$.

Hence, $\Delta f > 0$ for small h and k, and f has a local minimum value -1 at $(1, 1)$.