

# **MAT124 MATHEMATICS II**

## Multiple Integration

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Double Integrals

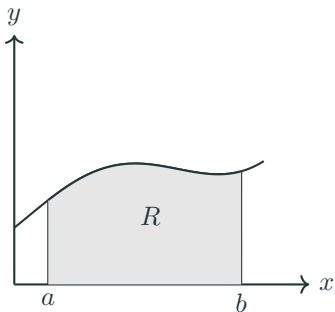
Properties of the Double Integral

Iteration of Double Integrals in Cartesian Coordinates

# Multiple Integration

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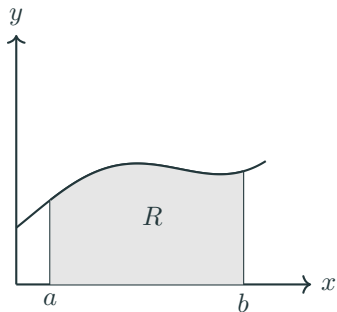
# Double Integrals



The area problem motivates the definition of the definite integral

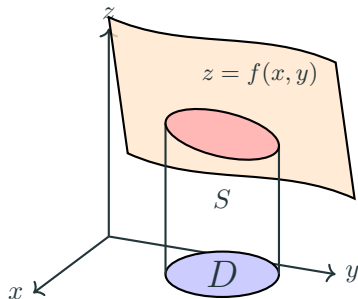
$$\int_a^b f(x) dx$$

# Double Integrals



The area problem motivates the definition of the definite integral

$$\int_a^b f(x) dx$$



The volume problem motivates the definition of the double integral

$$\iint_D f(x, y) dA$$

# Double Integrals

Let us start with the case where  $D$  is a closed rectangle with sides parallel to the coordinate axes in the  $xy$ -plane, and  $f$  is a bounded function on  $D$ . If  $D$  consists of the points  $(x, y)$  such that  $a \leq x \leq b$  and  $c \leq y \leq d$ , we can form a **partition**  $P$  of  $D$  into small rectangles by partitioning each of the intervals  $[a, b]$  and  $[c, d]$ , say by points

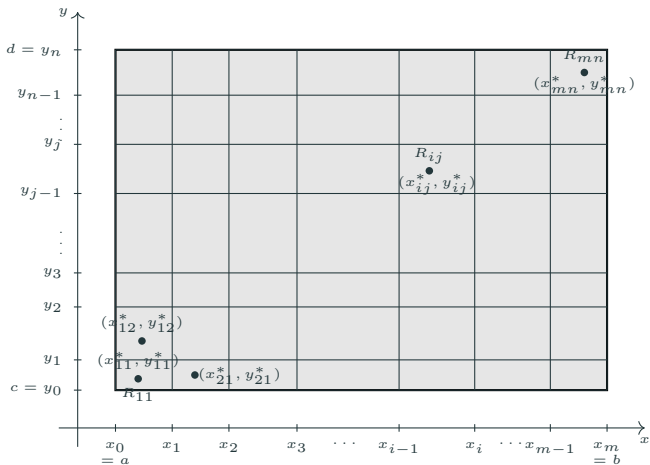
$$a = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = b,$$

$$c = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n = d.$$

# Double Integrals

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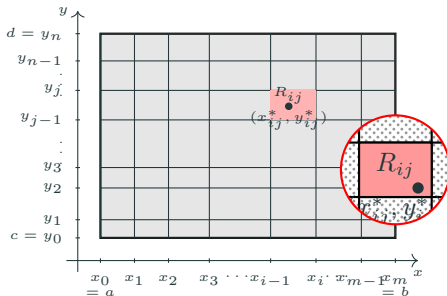
# Double Integrals

The rectangle  $R_{ij}$  has area

$$\Delta A_{ij} = \Delta x_i \Delta y_j = (x_i - x_{i-1})(y_j - y_{j-1})$$

and *diameter* (i.e., diagonal length)

$$\text{diam}(R_{ij}) = \sqrt{(\Delta x_i)^2 + (\Delta y_j)^2} = \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}.$$



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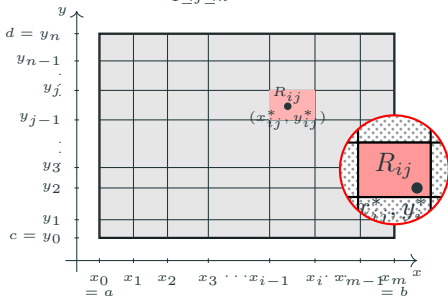
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$$\text{diam}(R_{ij}) = \sqrt{(\Delta x_i)^2 + (\Delta y_j)^2} = \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}.$$

The **norm** of the partition  $P$  is the largest of these subrectangle diameters:

$$\|P\| = \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \text{diam}(R_{ij}).$$

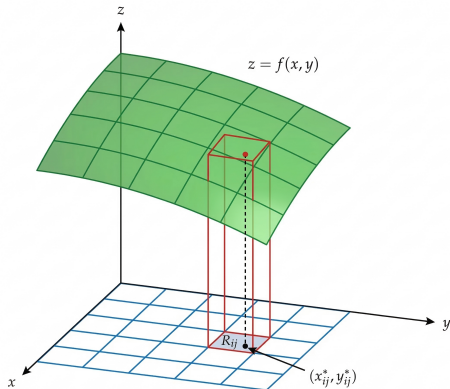


# Double Integrals

Now we pick an arbitrary point  $(x_{ij}^*, y_{ij}^*)$  in each of the rectangles  $R_{ij}$  and form the **Riemann sum**

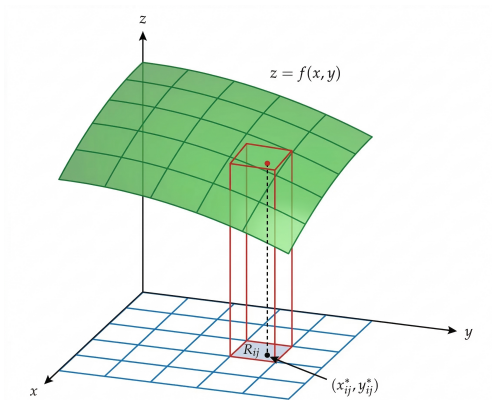
$$R(f, P) = \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A_{ij},$$

which is the sum of  $mn$  terms, one for each rectangle in the partition.



# Double Integrals

The double integral of  $f$  over  $D$  is defined to be the limit of such Riemann sums, provided the limit exists as  $\|P\| \rightarrow 0$  independently of how the points  $(x_{ij}^*, y_{ij}^*)$  are chosen.



# Double Integrals

Definition (The double integral over a rectangle)

We say that  $f$  is **integrable** over the rectangle  $D$  and has **double integral**

$$I = \iint_D f(x, y) dA,$$

if for every positive number  $\epsilon$  there exists a number  $\delta$  depending on  $\epsilon$ , such that

$$|R(f, P) - I| < \epsilon$$

holds for every partition  $P$  of  $D$  satisfying  $\|P\| < \delta$  and for all choices of the points  $(x_{ij}^*, y_{ij}^*)$  in the subrectangles of  $P$ .

## Double Integrals

### EXAMPLE

Let  $D$  be the square  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ . Use a Riemann sum corresponding to the partition of  $D$  into four smaller squares with points selected at the centre of each to find an approximate value for

$$\iint_D (x^2 + y) dA.$$

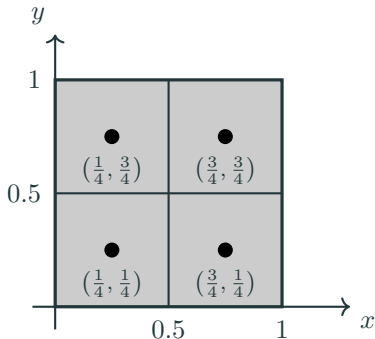
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### Solution:



The required partition  $P$  is formed by the lines  $x = 1/2$  and  $y = 1/2$ , which divide  $D$  into four squares, each of area  $\Delta A = 1/4$ . The centres of these squares are the points

$$\left(\frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{3}{4}, \frac{1}{4}\right), \text{ and } \left(\frac{3}{4}, \frac{3}{4}\right).$$

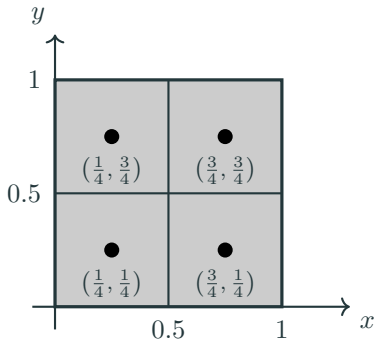
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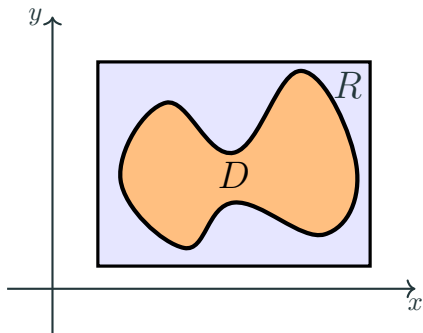
$$\iint_D (x^2 + y) dA.$$

**Solution:**



$$\begin{aligned} \iint_D (x^2 + y) dA &\approx R(x^2 + y, P) \\ &= \left(\frac{1}{16} + \frac{1}{4}\right) \frac{1}{4} + \left(\frac{1}{16} + \frac{3}{4}\right) \frac{1}{4} \\ &\quad + \left(\frac{9}{16} + \frac{1}{4}\right) \frac{1}{4} + \left(\frac{9}{16} + \frac{3}{4}\right) \frac{1}{4} \\ &= \frac{13}{16} = 0.8125. \end{aligned}$$

## Double Integrals over More General Domains



Bounded domain  $D$  is a subset of rectangle  $R$

## Double Integrals over More General Domains

If  $f(x, y)$  is defined and bounded on domain  $D$ , let  $\hat{f}$  be the extension of  $f$  that is zero everywhere outside  $D$ :

$$\hat{f}(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \text{ belongs to } D \\ 0, & \text{if } (x, y) \text{ does not belong to } D. \end{cases}$$

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If  $D$  is a *bounded* domain, then it is contained in some rectangle  $R$  with sides parallel to the coordinate axes. If  $\hat{f}$  is integrable over  $R$ , we say that  $f$  is **integrable** over  $D$  and define the **double integral** of  $f$  over  $D$  to be

$$\iint_D f(x, y) dA = \iint_R \hat{f}(x, y) dA.$$

## Double Integrals over More General Domains

### Theorem - Integrability of double variable functions

If  $f$  is continuous on a **closed, bounded** domain  $D$  whose boundary consists of finitely many curves of finite length, then  $f$  is integrable on  $D$ .

# Properties of the Double Integral

If  $f$  and  $g$  are integrable over  $D$ , and if  $L$  and  $M$  are constants, then

(a)  $\iint_D f(x, y) dA = 0$  if  $D$  has zero area.

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(c) **Integrals representing volumes:**

If  $f(x, y) \geq 0$  on  $D$ , then  $\iint_D f(x, y) dA = V \geq 0$ , where  $V$  is the volume of the solid lying vertically above  $D$  and below the surface  $z = f(x, y)$ .

## Properties of the Double Integral

(d) If  $f(x, y) \leq 0$  on  $D$ , then  $\iint_D f(x, y) dA = -V \leq 0$ , where  $V$  is the volume of the solid lying vertically below  $D$  and above the surface  $z = f(x, y)$ .

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(e) **Linear dependence on the integrand:**

$$\begin{aligned} \iint_D (Lf(x, y) + Mg(x, y)) dA \\ = L \iint_D f(x, y) dA + M \iint_D g(x, y) dA. \end{aligned}$$

## Properties of the Double Integral

(f) **Inequalities are preserved:**

If  $f(x, y) \leq g(x, y)$  on  $D$ , then  $\iint_D f(x, y) dA \leq \iint_D g(x, y) dA$ .

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(h) **Additivity of domains:** If  $D_1, D_2, \dots, D_k$  are nonoverlapping\* domains on each of which  $f$  is integrable, then  $f$  is integrable over the union  $D = D_1 \cup D_2 \cup \dots \cup D_k$  and

$$\iint_D f(x, y) dA = \sum_{j=1}^k \iint_{D_j} f(x, y) dA.$$

\*Nonoverlapping domains can share boundary points but have no interior points in common.

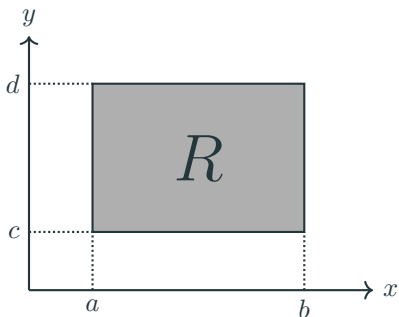
## Double Integrals by Inspection

### EXAMPLE

If  $R$  is the rectangle  $a \leq x \leq b$ ,  $c \leq y \leq d$ , then

$$\iint_R 3 \, dA = 3 \times \text{area of } R = 3(b-a)(d-c).$$

Here, the integrand is  $f(x, y) = 3$ , and the integral is equal to the volume of the solid box of height 3 whose base is the rectangle  $R$ .



The base of a rectangular box

## Double Integrals by Inspection

### EXAMPLE

Evaluate

$$I = \iint_{x^2+y^2 \leq 1} (\sin x + y^3 + 4) dA.$$

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**Solution:** The integral can be expressed as the sum of three integrals by property (e) of double integrals:

$$\begin{aligned} I &= \iint_{x^2+y^2 \leq 1} \sin x dA + \iint_{x^2+y^2 \leq 1} y^3 dA + \iint_{x^2+y^2 \leq 1} 4 dA \\ &= I_1 + I_2 + I_3. \end{aligned}$$

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The domain of integration is a circular disk of radius 1 centred at the origin. Since  $f(x, y) = \sin x$  is an *odd* function of  $x$ , its graph bounds as much volume below the  $xy$ -plane in the region  $x < 0$  as it does above the  $xy$ -plane in the region  $x > 0$ . These two contributions to the double integral cancel, so  $I_1 = 0$ .

Similarly,  $I_2 = 0$  because  $y^3$  is an odd function and  $D$  is symmetric about the  $x$ -axis.

## Double Integrals by Inspection

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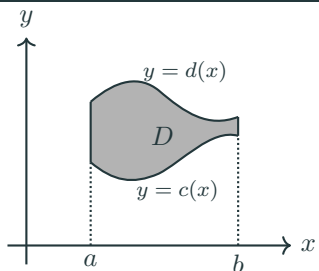
Finally,

$$I_3 = \iint_D 4 dA = 4 \times \text{area of } D = 4\pi.$$

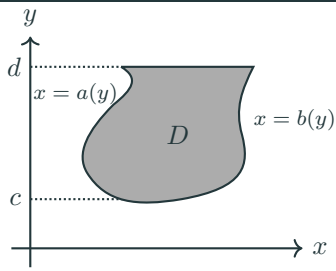
Thus,

$$I = 0 + 0 + 4\pi = 4\pi.$$

# Iteration of Double Integrals in Cartesian Coordinates



A  $y$ -simple domain



An  $x$ -simple domain

We say that the domain  $D$  in the  $xy$ -plane is  **$y$ -simple** if it is bounded by two vertical lines  $x = a$  and  $x = b$ , and two continuous graphs  $y = c(x)$  and  $y = d(x)$  between these lines. Lines parallel to the  $y$ -axis intersect a  $y$ -simple domain in an interval (possibly a single point) if at all. Similarly,  $D$  is  **$x$ -simple** if it is bounded by horizontal lines  $y = c$  and  $y = d$ , and two continuous graphs  $x = a(y)$  and  $x = b(y)$  between these lines.

# Iteration of Double Integrals in Cartesian Coordinates

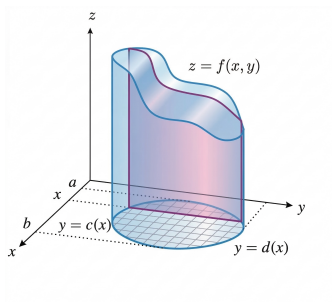
## Remark

Unlike the previous examples, most double integrals cannot be evaluated by inspection. We need a technique for evaluating double integrals similar to the technique for evaluating single definite integrals in terms of antiderivatives. Since the double integral represents a volume, we can evaluate it for simple domains by a slicing technique.

## Iteration of Double Integrals in Cartesian Coordinates

Suppose, for instance, that  $D$  is  $y$ -simple and is bounded by  $x = a$ ,  $x = b$ ,  $y = c(x)$ , and  $y = d(x)$ .

Then  $\iint_D f(x, y) dA$  represents (at least for positive  $f$ ) volume of the solid region inside the vertical cylinder through the boundary of  $D$  and between the  $xy$ -plane and the surface  $z = f(x, y)$ .

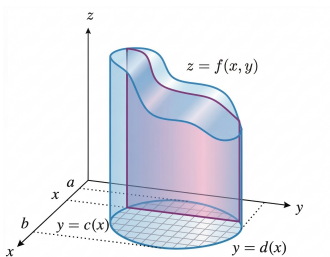


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Consider the cross-section of this solid in the vertical plane perpendicular to the  $x$ -axis at position  $x$ . Note that  $x$  is constant in that plane. If we use the projections of the  $y$ - and  $z$ -axes onto the plane as coordinate axes there, the cross-section is a plane region bounded by vertical lines  $y = c(x)$  and  $y = d(x)$ , by the horizontal line  $z = 0$ , and by the curve  $z = f(x, y)$ .

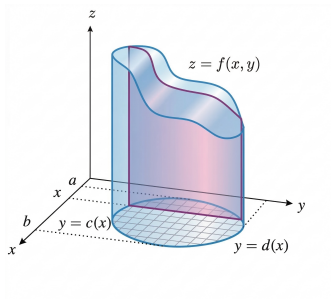


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Suppose, for instance, that  $D$  is  $y$ -simple and is bounded by  $x = a$ ,  $x = b$ ,  $y = c(x)$ , and  $y = d(x)$ .

The area of the cross-section is therefore given by

$$A(x) = \int_{c(x)}^{d(x)} f(x, y) dy.$$



# Iteration of Double Integrals in Cartesian Coordinates

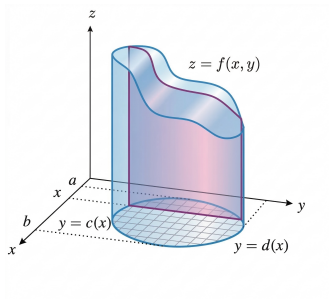
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$$\iint_D f(x, y) dA = \int_a^b A(x) dx$$

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# Iteration of Double Integrals in Cartesian Coordinates

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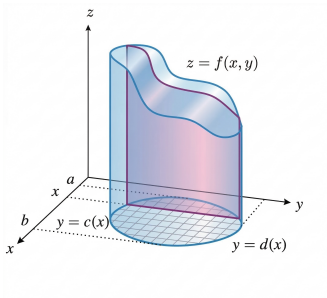
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$$= \int_a^b \left( \int_{c(x)}^{d(x)} f(x, y) dy \right) dx.$$

Notationally, it is common to omit the large parentheses and write

$$\iint_D f(x, y) dA = \int_a^b \int_{c(x)}^{d(x)} f(x, y) dy dx = \int_a^b dx \int_{c(x)}^{d(x)} f(x, y) dy.$$



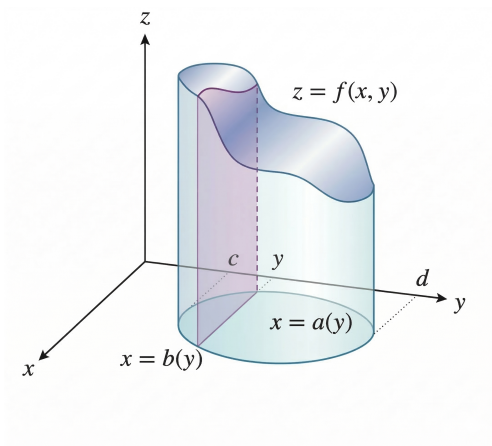
# Iteration of Double Integrals in Cartesian Coordinates

$$\int_a^b dx \underbrace{\int_{c(x)}^{d(x)} f(x, y) dy}_{\text{inner integral}}$$

$$\underbrace{\int_a^b \int_{c(x)}^{d(x)} f(x, y) dy dx}_{\text{iterated integrals}}$$

# Iteration of Double Integrals in Cartesian Coordinates

For double integrals over  $x$ -simple domains we can slice perpendicularly to the  $y$ -axis and obtain an iterated integral with the outer integral in the  $y$  direction.



# Iteration of Double Integrals in Cartesian Coordinates

## Theorem (Iteration of double integrals)

If  $f(x, y)$  is continuous on the bounded  $y$ -simple domain  $D$  given by  $a \leq x \leq b$  and  $c(x) \leq y \leq d(x)$ , then

$$\iint_D f(x, y) dA = \int_a^b dx \int_{c(x)}^{d(x)} f(x, y) dy.$$

Similarly, if  $f$  is continuous on the  $x$ -simple domain  $D$  given by  $c \leq y \leq d$  and  $a(y) \leq x \leq b(y)$ , then

$$\iint_D f(x, y) dA = \int_c^d dy \int_{a(y)}^{b(y)} f(x, y) dx.$$

# Iteration of Double Integrals in Cartesian Coordinates

## Remark

The three expressions

- $\iint_D f(x, y) dx dy,$
- $\iint_D f(x, y) dy dx,$
- $\iint_D f(x, y) dA$

all stand for the double integral of  $f$  over  $D$ .

Only when the double integral is iterated does the order of  $dx$  and  $dy$  become important.

# Iteration of Double Integrals in Cartesian Coordinates

## Example

### EXAMPLE

Find the volume of the solid lying above the square  $Q$  defined by  $0 \leq x \leq 1$  and  $1 \leq y \leq 2$  and below the plane  $z = 4 - x - y$ .

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**Solution:** The square  $Q$  is both *x-simple* and *y-simple*, so the double integral giving the volume can be iterated in either direction. We will do it both ways just for practice.

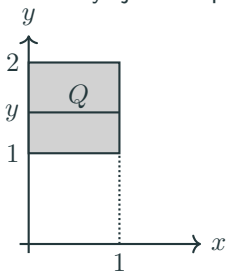
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**Solution:** The square  $Q$  is both  $x$ -simple and  $y$ -simple, so the double integral giving the volume can be iterated in either direction. We will do it both ways just for practice.



The horizontal line through  $Q$  indicates iteration with the inner integral in the  $x$  direction.

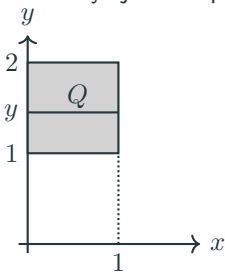
# Iteration of Double Integrals in Cartesian Coordinates

## Example

### EXAMPLE

Find the volume of the solid lying above the square  $Q$  defined by  $0 \leq x \leq 1$  and  $1 \leq y \leq 2$  and below the plane  $z = 4 - x - y$ .

**Solution:** The square  $Q$  is both  $x$ -simple and  $y$ -simple, so the double integral giving the volume can be iterated in either direction. We will do it both ways just for practice.



The horizontal line through  $Q$  indicates iteration with the inner integral in the  $x$  direction.

Volume above  $Q$  is  $\iint_Q (4 - x - y) dA$

$$\begin{aligned} &= \int_1^2 dy \int_0^1 (4 - x - y) dx \\ &= \int_1^2 dy \left( 4x - \frac{x^2}{2} - xy \right) \Big|_{x=0}^{x=1} \\ &= \int_1^2 \left( \frac{7}{2} - y \right) dy \\ &= \left( \frac{7y}{2} - \frac{y^2}{2} \right) \Big|_1^2 = 2 \text{ cubic units.} \end{aligned}$$

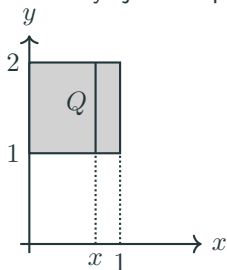
# Iteration of Double Integrals in Cartesian Coordinates

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The vertical line through  $Q$  indicates iteration with the inner integral in the  $y$  direction.

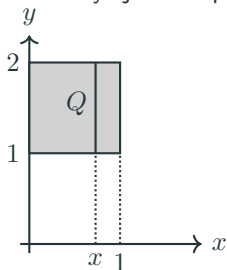
# Iteration of Double Integrals in Cartesian Coordinates

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The vertical line through  $Q$  indicates iteration with the inner integral in the  $y$  direction.

Volume above  $Q$  is  $\iint_Q (4 - x - y) dA$

$$\begin{aligned} &= \int_0^1 dx \int_1^2 (4 - x - y) dy \\ &= \int_0^1 dx \left( 4y - xy - \frac{y^2}{2} \right) \Big|_{y=1}^{y=2} \\ &= \int_0^1 \left( \frac{5}{2} - x \right) dx \\ &= \left( \frac{5x}{2} - \frac{x^2}{2} \right) \Big|_0^1 = 2 \text{ cubic units.} \end{aligned}$$

# Iteration of Double Integrals in Cartesian Coordinates

## Example

### EXAMPLE

Evaluate

$$\iint_T xy \, dA$$

over the triangle  $T$  with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(1, 1)$ .

# Iteration of Double Integrals in Cartesian Coordinates

## Example

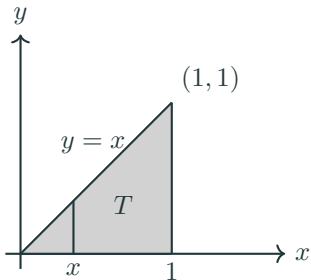
### EXAMPLE

Evaluate

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over the triangle  $T$  with vertices  $(0,0)$ ,  $(1,0)$ , and  $(1,1)$ .

**Solution:**



# Iteration of Double Integrals in Cartesian Coordinates

## Example

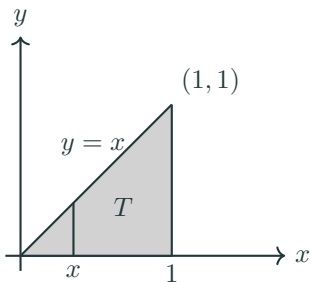
### EXAMPLE

Evaluate

$$\iint_T xy \, dA$$

over the triangle  $T$  with vertices  $(0,0)$ ,  $(1,0)$ , and  $(1,1)$ .

**Solution:**



$$\begin{aligned}\iint_T xy \, dA &= \int_0^1 dx \int_0^x xy \, dy \\ &= \int_0^1 dx \left( \frac{xy^2}{2} \right) \Bigg|_{y=0}^{y=x} \\ &= \int_0^1 \frac{x^3}{2} dx \\ &= \frac{x^4}{8} \Bigg|_0^1 = \frac{1}{8}.\end{aligned}$$

# Iteration of Double Integrals in Cartesian Coordinates

## Example

### EXAMPLE

Sketch and find the volume of the solid bounded by the planes  $y = 0$ ,  $z = 0$ , and  $z = a - x + y$  and the parabolic cylinder  $y = a - \left(\frac{x^2}{a}\right)$ , where  $a$  is a positive constant.

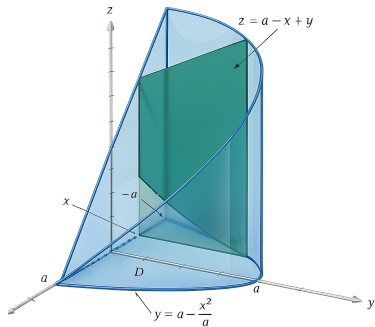
# Iteration of Double Integrals in Cartesian Coordinates

## Example

### EXAMPLE

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**Solution:** The volume is  $V = \iint_D (a - x + y) dA$



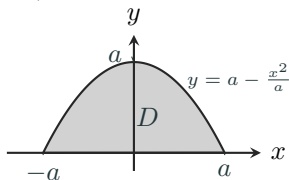
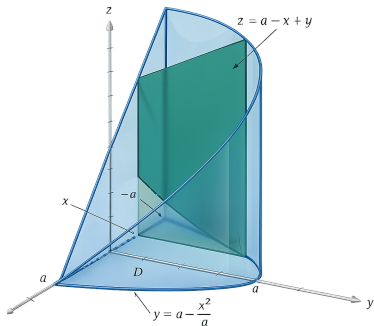
# Iteration of Double Integrals in Cartesian Coordinates

## Example

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**Solution:** The volume is  $V = \iint_D (a - x + y) dA$



By symmetry of the region  $D$  about the  $y$ -axis,

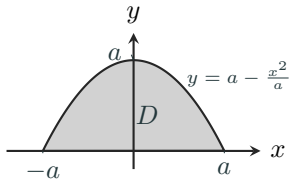
$$V = \iint_D (a + y) dA$$

# Iteration of Double Integrals in Cartesian Coordinates

Example

**Solution:**

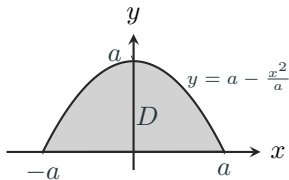
$$\begin{aligned} V &= \iint_D (a + y) dA \\ &= \int_{-a}^a dx \int_0^{a - (x^2/a)} (a + y) dy \end{aligned}$$



# Iteration of Double Integrals in Cartesian Coordinates

## Example

**Solution:**

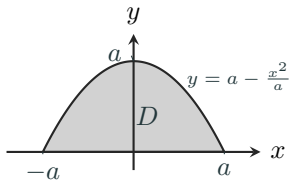


$$\begin{aligned} V &= \iint_D (a + y) \, dA \\ &= \int_{-a}^a dx \int_0^{a-(x^2/a)} (a + y) \, dy \\ &= \int_{-a}^a \left( ay + \frac{y^2}{2} \right) \Big|_{y=0}^{y=a-(x^2/a)} dx \\ &= \int_{-a}^a \left[ a^2 - x^2 + \frac{1}{2} \left( a^2 - 2x^2 + \frac{x^4}{a^2} \right) \right] dx \\ &= 2 \int_0^a \left[ \frac{3}{2}a^2 - 2x^2 + \frac{x^4}{2a^2} \right] dx \end{aligned}$$

# Iteration of Double Integrals in Cartesian Coordinates

## Example

**Solution:**



$$\begin{aligned}V &= \iint_D (a + y) \, dA \\&= \int_{-a}^a dx \int_0^{a-(x^2/a)} (a + y) \, dy \\&= \int_{-a}^a \left( ay + \frac{y^2}{2} \right) \Big|_{y=0}^{y=a-(x^2/a)} dx \\&= \int_{-a}^a \left[ a^2 - x^2 + \frac{1}{2} \left( a^2 - 2x^2 + \frac{x^4}{a^2} \right) \right] dx \\&= 2 \int_0^a \left[ \frac{3}{2}a^2 - 2x^2 + \frac{x^4}{2a^2} \right] dx \\&= \left( 3a^2x - \frac{4x^3}{3} + \frac{x^5}{5a^2} \right) \Big|_0^a \\&= 3a^3 - \frac{4}{3}a^3 + \frac{1}{5}a^3 = \frac{28}{15}a^3 \text{ cubic units.}\end{aligned}$$