MAT124 MATHEMATICS II

Langrange Multipliers

Outline

Langrange Multipliers

Problems with More than One Constraint

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Objective: dealing with problems of maximizing or minimizing functions whose variables are not independent of one another.

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THEOREM

Suppose that f and g have continuous first partial derivatives near the point $P_0 = (x_0, y_0)$ on the curve $\mathcal{C}$ with equation $g(x, y) = 0$. Suppose also that, when restricted to points on $\mathcal{C}$, the function $f(x, y)$ has a local maximum or minimum value at P_0. Finally, suppose that

(i) P_0 is not an endpoint of $\mathcal{C}$, and

(ii) $\nabla g(P_0) \neq \mathbf{0}$.

Then there exists a number λ_0 such that (x_0, y_0, λ_0) is a critical point of the Lagrangian function

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y).$$

Langrange Multipliers

To find candidates for points on the curve $g(x, y) = 0$ at which $f(x, y)$ is maximum or minimum, we should look for critical points of the Lagrangian function

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y).$$

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$$\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda g(x,y).$$

At any critical point of L we must have

$$\left.\begin{aligned} 0 &= \frac{\partial L}{\partial x} = f_1(x,y) + \lambda g_1(x,y), \\ 0 &= \frac{\partial L}{\partial y} = f_2(x,y) + \lambda g_2(x,y), \end{aligned}\right\} \quad \text{that is, } \nabla f \text{ is parallel to } \nabla g,$$

$$\text{and} \quad \left. 0 = \frac{\partial L}{\partial \lambda} = g(x,y), \right\} \qquad \text{the constraint equation.}$$

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Note, however, that it is assumed that the constrained problem **has a solution.**

Langrange Multipliers - Example

EXAMPLE

Find the shortest distance from the origin to the curve $x^2y = 16$.

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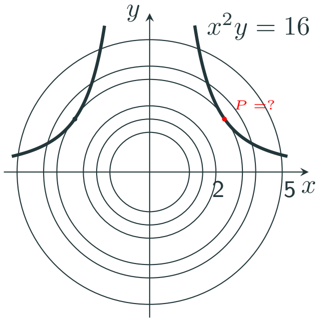
Solution:

PROBLEM: minimize $f(x, y) = x^2 + y^2$

subject to $g(x, y) = x^2y - 16 = 0.$



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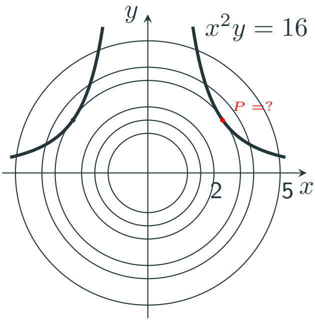

Let $L(x, y, \lambda) = x^2 + y^2 + \lambda(x^2y - 16)$.

$$0 = \frac{\partial L}{\partial x} = 2x + 2\lambda xy = 2x(1 + \lambda y)$$

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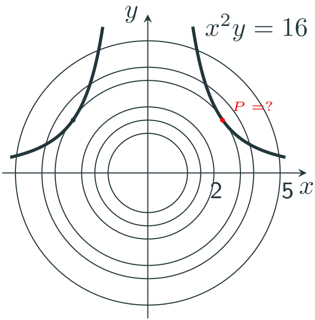

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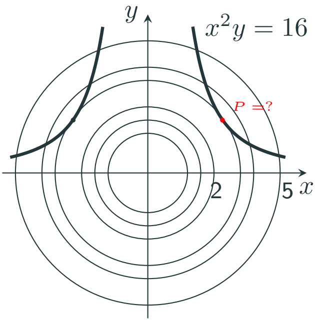

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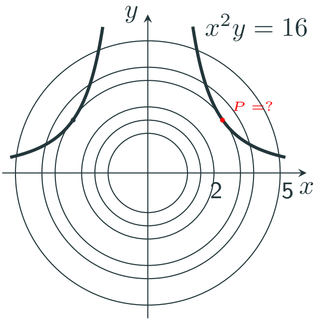

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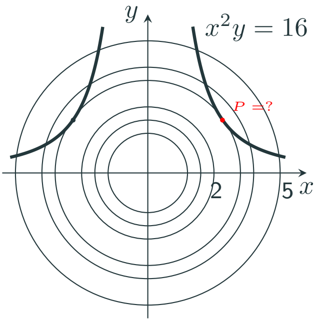

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$$0 = \frac{\partial L}{\partial \lambda} = x^2y - 16$$

Since $x^2y - 16 = 0$ and $2y^2 - x^2 = 0$, we get

$$2y^3 = 16 \implies y = 2$$

Langrange Multipliers - Example

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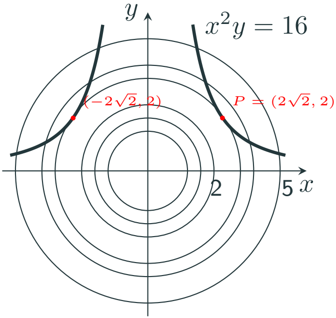

When $y = 2$, we have $x = \pm 2\sqrt{2}$. Finally,

candidates for points on $x^2y = 16$
closest to the origin: $(\pm 2\sqrt{2}, 2)$

the minimum distance $= 2\sqrt{3}$

Langrange Multipliers - Example

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Let $L(x, y, \lambda) = x^2 + y^2 + \lambda(17x^2 + 12xy + 8y^2 - 100)$, we have

$$
\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 2x + \lambda(34x + 12y) \\
0 &= \frac{\partial L}{\partial y} = 2y + \lambda(12x + 16y) \\
0 &= \frac{\partial L}{\partial \lambda} = 17x^2 + 12xy + 8y^2 - 100
\end{aligned}
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By first two equations, we have

$$\frac{-2x}{34x + 12y} = \frac{-2y}{12x + 16y} \Rightarrow 2x^2 - 3xy - 2y^2 = 0$$

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Find the points on the curve $17x^2 + 12xy + 8y^2 = 100$ that are closest to and farthest away from the origin.

Solution:

$$0 = \frac{\partial L}{\partial \lambda} = 17x^2 + 12xy + 8y^2 - 100$$

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$\times 4$

$$25x^2 = 100$$

Langrange Multipliers - Example

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For $x = 2$:

$$y^2 + 3y - 4 = (y-1)(y+4) = 0,$$

For $x = -2$:

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$$y^2 + 3y - 4 = (y-1)(y+4) = 0,$$

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candidate points: $(2,1), (-2,-1)$,(2,-4), and $(-2,4)$.

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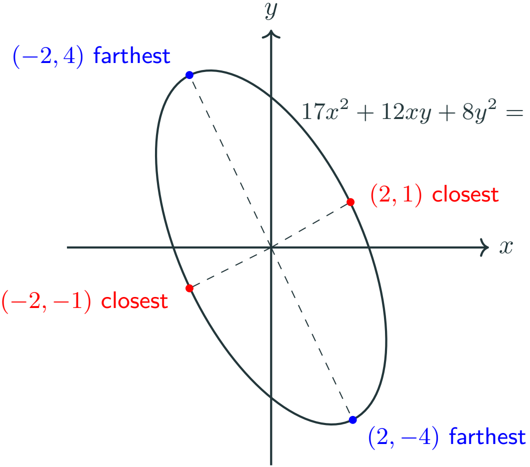

Langrange Multipliers - Example

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The Lagrangian function is $L(x, y, \lambda) = xy + \lambda\left(\dfrac{x^2}{8} + \dfrac{y^2}{2} - 1\right)$ and its critical points are given by

$$
\begin{aligned}
0 &= \frac{\partial L}{\partial x} = y + \lambda\frac{x}{4}, \\
0 &= \frac{\partial L}{\partial y} = x + \lambda y, \\
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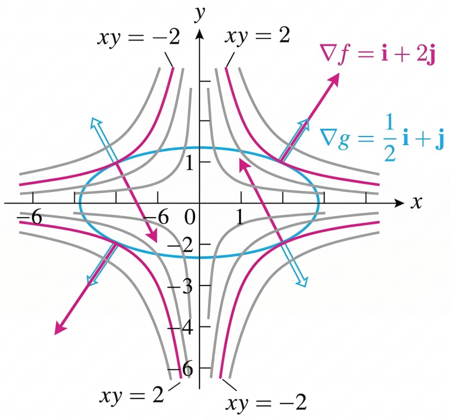
Find the greatest and smallest values that the function $f(x, y) = xy$ takes on the ellipse $\dfrac{x^2}{8} + \dfrac{y^2}{2} = 1$.

Solution: The geometry of the solution also guaranties the existence of the solution:



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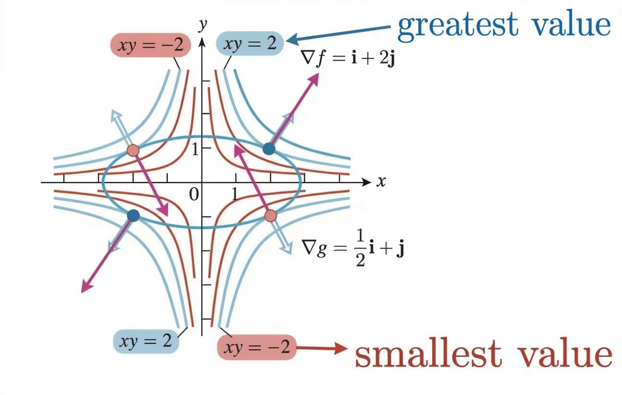

Langrange Multipliers - Example

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Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

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Solution: We want to extremize the function $f(x, y) = 3x + 4y$ subject to $g(x, y) = x^2 + y^2 - 1 = 0$.

The Lagrangian is

$$L(x, y, \lambda) = 3x + 4y + \lambda(x^2 + y^2 - 1).$$

$$0 = \frac{\partial L}{\partial x} = 3 + 2\lambda x$$
$$0 = \frac{\partial L}{\partial y} = 4 + 2\lambda y$$
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Langrange Multipliers - Example

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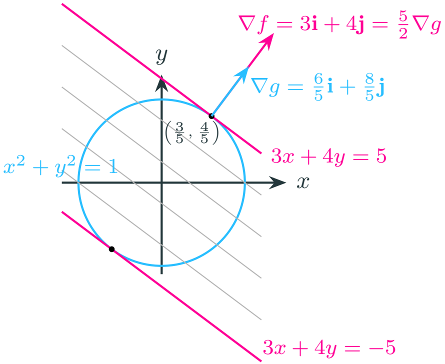
Solution: The geometry of the solution:



The function $f(x, y) = 3x + 4y$ takes on its largest value on the unit circle

$$g(x, y) = x^2 + y^2 - 1 = 0$$

at the point $(3/5,\, 4/5)$ and its smallest value at the point $(-3/5,\, -4/5)$.

Extremize $f(x, y, z)$ subject to $g(x, y, z) = 0$ and $h(x, y, z) = 0$

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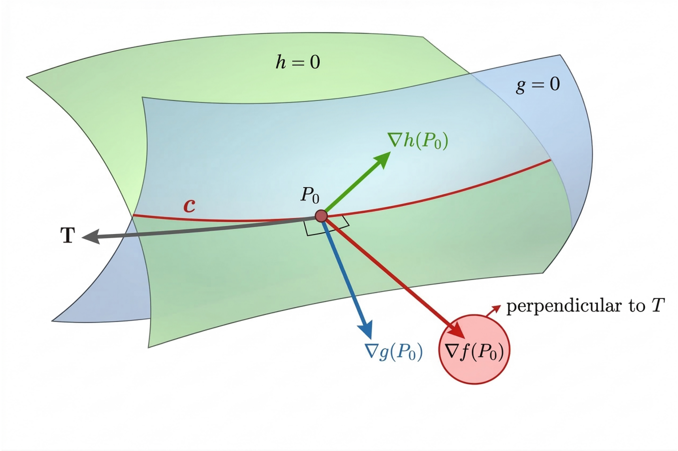
Again, we assume that the problem has a solution, say, at the point $P_0 = (x_0, y_0, z_0)$, and that the functions f, g, and h have continuous first partial derivatives near P_0. Also, we assume that $\mathbf{T} = \nabla g(P_0) \times \nabla h(P_0) \neq \mathbf{0}$. These conditions imply that the surfaces $g(x, y, z) = 0$ and $h(x, y, z) = 0$ are smooth near P_0 and are not tangent to each other there, so they must intersect in a curve $\mathcal{C}$ that is smooth near P_0. The curve $\mathcal{C}$ has tangent vector $\mathbf{T}$ at P_0.

Langrange Multipliers

Problems with More than One Constraint



Extremize $f(x, y, z)$ subject to $g(x, y, z) = 0$ and $h(x, y, z) = 0$

Since $\nabla g(P_0)$ and $\nabla h(P_0)$ are nonzero and both are perpendicular to $\mathbf{T}$, $\nabla f(P_0)$ must lie in the plane spanned by these two vectors and hence must be a linear combination of them:

$$\nabla f(x_0, y_0, z_0) = -\lambda_0 \nabla g(x_0, y_0, z_0) - \mu_0 \nabla h(x_0, y_0, z_0)$$

for some constants λ_0 and μ_0. It follows that $(x_0, y_0, z_0, \lambda_0, \mu_0)$ is a critical point of the Lagrangian function

$$L(x, y, z, \lambda, \mu) = f(x, y, z) + \lambda g(x, y, z) + \mu h(x, y, z).$$

We look for triples (x, y, z) that extremize $f(x, y, z)$ subject to the two constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$ among the points (x, y, z, λ, μ) that are critical points of the above Lagrangian function,

$$L(x, y, z, \lambda, \mu) = f(x, y, z) + \lambda g(x, y, z) + \mu h(x, y, z).$$

and we therefore solve the system of equations

$$\begin{aligned} f_1(x, y, z) + \lambda g_1(x, y, z) + \mu h_1(x, y, z) &= 0, \\ f_2(x, y, z) + \lambda g_2(x, y, z) + \mu h_2(x, y, z) &= 0, \\ f_3(x, y, z) + \lambda g_3(x, y, z) + \mu h_3(x, y, z) &= 0, \\ g(x, y, z) &= 0, \\ h(x, y, z) &= 0. \end{aligned}$$

Problems with More than One Constraint - Example

EXAMPLE

Find the maximum and minimum values of $f(x, y, z) = xy + 2z$ on the circle that is the intersection of the plane $x + y + z = 0$ and the sphere $x^2 + y^2 + z^2 = 24$.

Problems with More than One Constraint - Example

EXAMPLE

Find the maximum and minimum values of $f(x, y, z) = xy + 2z$ on the circle that is the intersection of the plane $x + y + z = 0$ and the sphere $x^2 + y^2 + z^2 = 24$.

Solution: The function f is continuous, and the intersection circle is a closed bounded set in 3-space. Therefore, maximum and minimum values must exist.

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$$L = xy + 2z + \lambda(x + y + z) + \mu(x^2 + y^2 + z^2 - 24).$$

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Setting the first partial derivatives of L equal to zero, we obtain

$$\begin{aligned} y + \lambda + 2\mu x &= 0, \\ x + \lambda + 2\mu y &= 0, \\ 2 + \lambda + 2\mu z &= 0, \\ x + y + z &= 0, \\ x^2 + y^2 + z^2 - 24 &= 0. \end{aligned}$$

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$$\left.\begin{aligned} y + \lambda + 2\mu x &= 0 \\ x + \lambda + 2\mu y &= 0, \end{aligned}\right\} \longrightarrow (x - y)(1 - 2\mu) = 0 \longrightarrow \mu = \tfrac{1}{2} \text{ or } x = y.$$

$$\begin{aligned} 2 + \lambda + 2\mu z &= 0, \\ x + y + z &= 0, \\ x^2 + y^2 + z^2 - 24 &= 0. \end{aligned}$$

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$\boxed{z = -1 \quad \text{and} \quad x + y = 1}$

$\rightarrow \boxed{x^2 + y^2 = 24 - z^2 = 23}$

$x^2 + y^2 + 2xy = (x + y)^2 = 1 \rightarrow 2xy = 1 - 23 = -22$

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$x^2 + y^2 + 2xy = (x + y)^2 = 1 \rightarrow 2xy = 1 - 23 = -22$

$xy = -11$ $\quad (x - y)^2 = x^2 + y^2 - 2xy = 23 + 22$

$x - y = \pm 3\sqrt{5}$

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CASE I: Since $x - y = \pm 3\sqrt{5}$ and $x + y = 1$, we have

$$\left(\frac{1 + 3\sqrt{5}}{2}, \frac{1 - 3\sqrt{5}}{2}, -1\right)$$

$$\left(\frac{1 - 3\sqrt{5}}{2}, \frac{1 + 3\sqrt{5}}{2}, -1\right)$$

At both of these points we find that

$$f(x, y, z) = xy + 2z = -11 - 2 = -13.$$

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CASE II: $x = y$

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Therefore, points $(2, 2, -4)$ and $(-2, -2, 4)$ must be considered.

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$$\boxed{z = -2x} \longrightarrow \boxed{6x^2 = 24} \longrightarrow x = \pm 2.$$

Therefore, points $(2, 2, -4)$ and $(-2, -2, 4)$ must be considered.

$$f(2, 2, -4) = 4 - 8 = -4$$
$$f(-2, -2, 4) = 4 + 8 = 12.$$

We conclude that the maximum value of f on the circle is 12, and the minimum value is -13.

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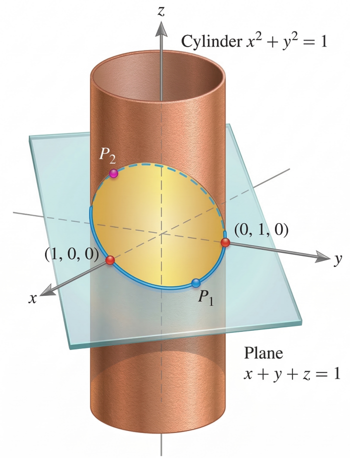
The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.



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The method of Lagrange multipliers can be applied to find extreme values of a function of n variables, that is, of a vector variable $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ subject to $m \le n - 1$ constraints:

Extremize $f(\mathbf{x})$ **subject to** $g_{(1)}(\mathbf{x}) = 0, \ \ldots \ , \ g_{(m)}(\mathbf{x}) = 0.$

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Assuming that the problem has a solution at the point P_0, that f and all of the functions $g_{(j)}$ have continuous first partial derivatives in a neighbourhood of P_0, and that the intersection of the constraint (hyper)surfaces is smooth near P_0, then we should look for P_0 among the critical points of the $(n+m)$-variable Lagrangian function

$$L(\mathbf{x}, \lambda_1, \ldots, \lambda_m) = f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j g_{(j)}(\mathbf{x}).$$

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Any critical points must satisfy the $n+m$ equations

$$\frac{\partial L}{\partial x_i} = 0, \quad (1 \le i \le n), \qquad \frac{\partial L}{\partial \lambda_j} = g_{(j)}(\mathbf{x}) = 0, \quad (1 \le j \le m).$$