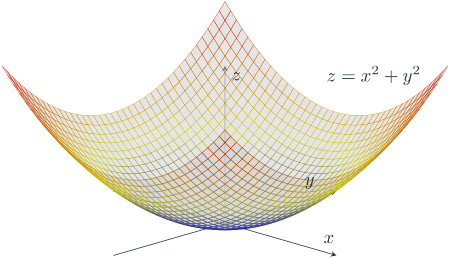
MAT124 MATHEMATICS II

Applications of Partial Derivatives - Extreme Values

Outline

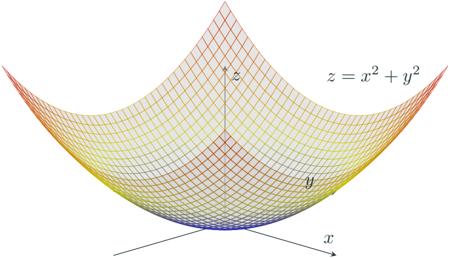
Applications of Partial Derivatives

Extreme Values



$x^2 + y^2$ has minimum value 0 at the origin

Extreme Values



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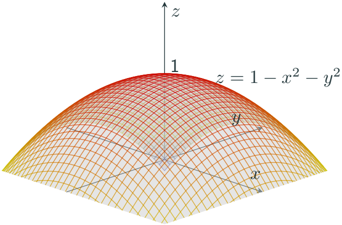

$1 - x^2 - y^2$ has maximum value 1 at the origin

Extreme Values

Extreme (Local - Absolute) Values

We say that a function of two variables has a local maximum value at the point (a, b) in its domain if $f(x, y) \leq f(a, b)$ for all points (x, y) in the domain of f that are sufficiently close to the point (a, b). If the inequality holds for all (x, y) in the domain of f, then we say that f has an absolute maximum value at (a, b).

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Similar definitions obtain for local and absolute minimum values. In practice, the word absolute is usually omitted, and we refer simply to the maximum or the minimum value of f.

Extreme Values

Theorem (Necessary conditions for extreme values)

A function $f(x, y)$ can have a local or absolute extreme value at a point (a, b) in its domain only if (a, b) is one of the following:

(a) a critical point of f, that is, a point satisfying $\nabla f(a, b) = 0$,

(b) a singular point of f, that is, a point where $\nabla f(a, b)$ does not exist, or

(c) a boundary point of the domain of f.

Extreme Values

A set in $\mathbb{R}^n$ is bounded if it is contained inside some *ball* $x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2$ of finite radius R. A set on the real line is bounded if it is contained in an interval of finite length.

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Theorem (Sufficient conditions for extreme values)

If f is a continuous function of n variables whose domain is a closed and bounded set in $\mathbb{R}^n$, then the range of f is a bounded set of real numbers, and there are points in its domain where f takes on absolute maximum and minimum values.

Extreme Values Examples

EXAMPLE

- The function $f(x, y) = x^2 + y^2$ has a critical point at $(0, 0)$, since $\nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j}$ and both components of ∇f vanish at $(0, 0)$. Since

$$f(x, y) > 0 = f(0, 0) \quad \text{if} \quad (x, y) \neq (0, 0),$$

f must have (absolute) minimum value 0 at that point. If the domain of f is not restricted, f has no maximum value.

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f must have (absolute) minimum value 0 at that point. If the domain of f is not restricted, f has no maximum value.

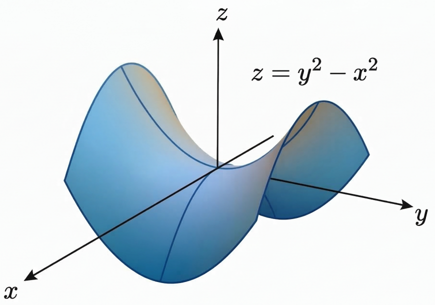
- Similarly, since $\nabla g = -2x\,\mathbf{i} - 2y\,\mathbf{j}$

$$g(x, y) = 1 - x^2 - y^2$$

has (absolute) maximum value 1 at its critical point $(0, 0)$.

EXAMPLE

The function $h(x,y) = y^2 - x^2$ also has a critical point at $(0,0)$ but has neither a local maximum nor a local minimum value at that point. Observe that $h(0,0) = 0$ but $h(x,0) < 0$ and $h(0,y) > 0$ for all nonzero values of x and y. The graph of h is a hyperbolic paraboloid. In view of the shape of this surface, we call the critical point $(0,0)$ a saddle point of h.



Extreme Values - Saddle point

In general, we will somewhat loosely call any interior critical point of the domain of a function f of several variables a saddle point if f does not have a local maximum or minimum value there.

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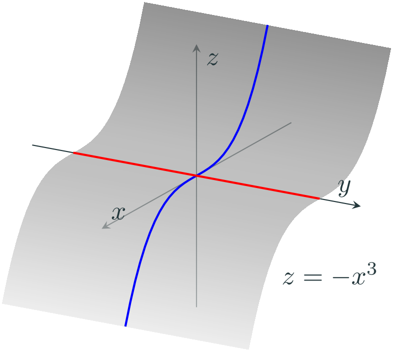
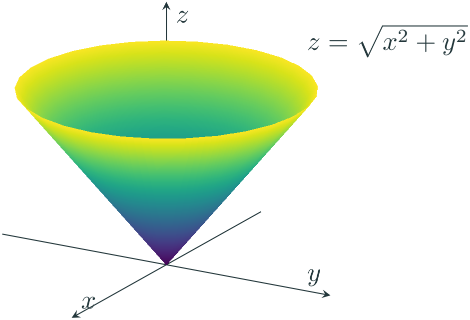
Near a saddle point, the graph will not always look like a saddle.



EXAMPLE

The function $f(x, y) = \sqrt{x^2 + y^2}$ has no critical points but does have a singular point at $(0, 0)$ where it has a local (and absolute) minimum value, zero. The graph of f is (one nappe of) a circular cone.



Extreme Values Examples

EXMAPLE

The function $f(x, y) = 1 - x$ is defined everywhere in the xy-plane and has no critical or singular points. ($\nabla f(x, y) = -\mathbf{i}$ at every point (x, y).) Therefore f has no extreme values.

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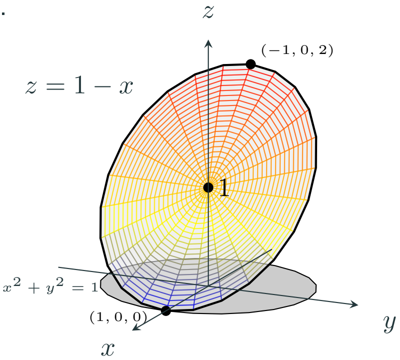

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Solution: The critical points must satisfy the system of equations:

$$0 = f_1(x, y) = 6x^2 - 6y \qquad \Longleftrightarrow \qquad x^2 = y$$
$$0 = f_2(x, y) = -6x + 6y \qquad \Longleftrightarrow \qquad x = y.$$

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Consider the point $(0, 0)$. Let Δf be

$$\Delta f = f(h, k) - f(0, 0) = 2h^3 - 6hk + 3k^2.$$

Since $f(h, 0) - f(0, 0) = 2h^3$ is positive for small positive h and negative for small negative h, f cannot have a maximum or minimum value at $(0, 0)$. Therefore, $(0, 0)$ is a saddle point.

EXMAPLE

Find and classify the critical points of $f(x, y) = 2x^3 - 6xy + 3y^2$.

Solution: Now consider $(1, 1)$. Let Δf be

$$\begin{aligned}\Delta f &= f(1+h, 1+k) - f(1, 1) \\ &= 2(1+h)^3 - 6(1+h)(1+k) + 3(1+k)^2 - (-1) \\ &= 2 + 6h + 6h^2 + 2h^3 - 6 - 6h - 6k - 6hk + 3 + 6k + 3k^2 + 1 \\ &= 6h^2 - 6hk + 3k^2 + 2h^3 \\ &= 3(h-k)^2 + h^2(3+2h).\end{aligned}$$

Both terms in the latter expression are nonnegative if $|h| < 3/2$, and they are not both zero unless $h = k = 0$.

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Both terms in the latter expression are nonnegative if $|h| < 3/2$, and they are not both zero unless $h = k = 0$.

Hence, $\Delta f > 0$ for small h and k, and f has a local minimum value -1 at $(1,1)$.

Extreme Values

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij},$$

where A_{ij} is the $(n-1) \times (n-1)$ determinant obtained by deleting the ith row and jth column from $\mathbf{A}$.

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where A_{ij} is the $(n-1) \times (n-1)$ determinant obtained by deleting the ith row and jth column from $\mathbf{A}$.

Note that this is the expansion along the ith row. It is possible to get expansion along the jth column as

$$\det(\mathbf{A}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} A_{ij}.$$

EXAMPLE

Here is the expansion of a certain 4×4 determinant about its third column:

$$\begin{vmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 0 & 0 & 2 \\ -1 & 1 & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 2 & 1 & 1 \\ 3 & 0 & 2 \\ -1 & 1 & 0 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= -\left(-3 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} \right) - (-1) \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 3(0-1) + 2(2+1) + 1(2-3) = 2.$$

Extreme Values

If $\mathbf{x}$ is a column vector in $\mathbb{R}^n$ and $\mathbf{A} = [a_{ij}]$ is an $n \times n$, real, symmetric matrix (i.e., $a_{ij} = a_{ji}$ for $1 \leq i, j \leq n$), then the expression

$$Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

is called a quadratic form on $\mathbb{R}^n$ corresponding to the matrix $\mathbf{A}$. Observe that $Q(\mathbf{x})$ is a real number for any vector $\mathbf{x}$.

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We say that the matrix $\mathbf{A}$ is positive definite if $Q(\mathbf{x}) > 0$ for every nonzero vector $\mathbf{x}$. Similarly, the matrix $\mathbf{A}$ is negative definite if $Q(\mathbf{x}) < 0$ for every nonzero vector $\mathbf{x}$.

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If $Q(\mathbf{x}) > 0$ for some nonzero vectors $\mathbf{x}$ while $Q(\mathbf{x}) < 0$ for other such, then we will say that $\mathbf{A}$ is indefinite.

EXAMPLE

The expression $Q(x, y, z) = 3x^2 + 2y^2 + 5z^2 - 2xy + 4xz + 2yz$ is a quadratic form on $\mathbb{R}^3$ corresponding to the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix}.$$

The matrix $\mathbf{A}$ is positive definite since $Q(x, y, z)$ can be rewritten in the form

$$Q(x, y, z) = x^2 + (x - y)^2 + (x + 2z)^2 + (y + z)^2,$$

from which it is apparent that $Q(x, y, z) \geq 0$ for all (x, y, z) and $Q(x, y, z) = 0$ only if $x = y = z = 0$.

Extreme Values

Theorem

Let $\mathbf{A} = [a_{ij}]_{i,j=1}^{n}$ be a real symmetric matrix and consider the determinants

$$D_i = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} \end{vmatrix} \qquad \text{for } 1 \le i \le n.$$

Thus, $D_1 = a_{11}$, $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} - a_{12}^2$, etc.

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(a) If $D_i > 0$ for $1 \leq i \leq n$, then $\mathbf{A}$ is positive definite.

(b) If $D_i > 0$ for even numbers i in $\{1, 2, \dots, n\}$, and $D_i < 0$ for odd numbers i in $\{1, 2, \dots, n\}$, then $\mathbf{A}$ is negative definite.

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(b) If $D_i > 0$ for even numbers i in $\{1, 2, \ldots, n\}$, and $D_i < 0$ for odd numbers i in $\{1, 2, \ldots, n\}$, then $\mathbf{A}$ is negative definite.

(c) If $\det(\mathbf{A}) = D_n \neq 0$ but neither of the above conditions hold, then $Q(x)$ is indefinite.

Theorem - A second derivative test

Suppose that $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a critical point of $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ and is interior to the domain of f. Also, suppose that all the second partial derivatives of f are continuous throughout a neighbourhood of $\mathbf{a}$, so that the Hessian matrix

$$\mathcal{H}(\mathbf{x}) = \begin{bmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \cdots & f_{1n}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) & \cdots & f_{2n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(\mathbf{x}) & f_{n2}(\mathbf{x}) & \cdots & f_{nn}(\mathbf{x}) \end{bmatrix}$$

is also continuous in that neighbourhood. Note that the continuity of the partials guarantees that $\mathcal{H}$ is a symmetric matrix.

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For the critical point $\mathbf{a} = (a_1, a_2, \ldots, a_n)$:

(a) If $\mathcal{H}(\mathbf{a})$ is positive definite, then f has a local minimum at $\mathbf{a}$.

(b) If $\mathcal{H}(\mathbf{a})$ is negative definite, then f has a local maximum at $\mathbf{a}$.

(c) If $\mathcal{H}(\mathbf{a})$ is indefinite, then f has a saddle point at $\mathbf{a}$.

(d) If $\mathcal{H}(\mathbf{a})$ is neither positive nor negative definite nor indefinite, this test gives no information.

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Find and classify the critical points of the function
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Solution:

- **Find critical points:**

$$0 = f_1(x,y,z) = 2xy - 2, \tag{1}$$

$$0 = f_2(x,y,z) = x^2 + 2yz, \tag{2}$$

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By equation (3), we have $-(y^2)/2 = z$. By writing this in equation (2), we get $y^3 = x^2$. By combining this and equation (1), $x^{5/3} = 1$.

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By equation (3), we have $-(y^2)/2 = z$. By writing this in equation (2), we get $y^3 = x^2$. By combining this and equation (1), $x^{5/3} = 1$. The only critical point is $P = (1,\ 1,\ -\frac{1}{2})$.

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$f(x, y, z) = x^2y + y^2z + z^2 - 2x.$

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- **Apply second derivative test:** The Hessian matrix is

$$\mathcal{H} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

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Since

$$D_1 = 2 > 0, D_2 = \begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} = -6 < 0, \; D_3 = \begin{vmatrix} 2 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 2 \end{vmatrix} = -20 < 0,$$

$\mathcal{H}$ is indefinite, so P is a saddle point of f.

Suppose that (a, b) is a critical point of the function $f(x, y)$ that is interior to the domain of f. Suppose also that the second partial derivatives of f are continuous in a neighbourhood of (a, b) and have at that point the values

$$A = f_{11}(a, b), \qquad B = f_{12}(a, b) = f_{21}(a, b), \qquad \text{and} \qquad C = f_{22}(a, b).$$

(a) If $B^2 - AC < 0$ and $A > 0$, then f has a local minimum value at (a, b).

(b) If $B^2 - AC < 0$ and $A < 0$, then f has a local maximum value at (a, b).

(c) If $B^2 - AC > 0$, then f has a saddle point at (a, b).

(d) If $B^2 - AC = 0$, this test provides no information; f may have a local maximum or a local minimum value or a saddle point at (a, b).

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$$f_1(x, y) = y(1 - x^2)e^{-(x^2+y^2)/2} = 0,$$
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So

critical points
$(0,0),\ (1,1),\ (1,-1),$
$(-1,1),$ and $(-1,-1)$

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- **Find critical points:**

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$$f_2(x,y) = x(1-y^2)e^{-(x^2+y^2)/2} = 0,$$

So

critical points
$(0,0),\ (1,1),\ (1,-1),$
$(-1,1),$ and $(-1,-1)$

- **Find second order partial derivatives:**

$$f_{11}(x,y) = xy(x^2-3)e^{-(x^2+y^2)/2},$$

$$f_{12}(x,y) = (1-x^2)(1-y^2)e^{-(x^2+y^2)/2},$$

$$f_{22}(x,y) = xy(y^2-3)e^{-(x^2+y^2)/2}.$$

Extreme Values Example

EXAMPLE

Find and classify the critical points of $f(x, y) = xye^{-(x^2+y^2)/2}$.

Solution:

- At $(0, 0)$: We have $A = C = 0$, $B = 1$, so that $B^2 - AC = 1 > 0$. Thus, f has a saddle point at $(0, 0)$.

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- At $(1,-1)$ and $(-1,1)$: 'e have $A = C = 2/e > 0$, $B = 0$. It follows that $B^2 - AC = -4/e^2 < 0$. Thus, f has local minimum values at these points. The value of f at each of them is $-1/e$.

Extreme Values of Functions Defined on Restricted Domains

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If f is a continuous function of n variables whose domain is a closed and bounded set in $\mathbb{R}^n$, then the range of f is a bounded set of real numbers, and there are points in its domain where f takes on absolute maximum and minimum values.

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1. List the interior points of R where f may have local maxima and minima and evaluate f at these points. These are the critical and/or singular points of f.

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2. List the boundary points of R where f has local maxima and minima and evaluate f at these points.
3. Look through the lists for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R.

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$$f_1(x, y) = 2y \qquad \text{and} \qquad f_2(x, y) = 2x,$$

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Boundary points of the closed disk is the circle $x^2 + y^2 = 4$. And one of its parametric representations is

$$x = 2\cos t,\ y = 2\sin t \qquad (-\pi \leq t \leq \pi)$$

EXAMPLE

Find the maximum and minimum values of $f(x, y) = 2xy$ on the closed disk $x^2 + y^2 \leq 4$.

Solution: So

$$f(2\cos t, 2\sin t) = 8\cos t \sin t = g(t).$$

So we should find absolute maximum and minimum values of single variable function g over closed interval $[-\pi, \pi]$.

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g has max. value 4 and min. value -4. Finally,

$\therefore$ f has the absolute max. value 4 at $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$

and the absolute min. value -4 at $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

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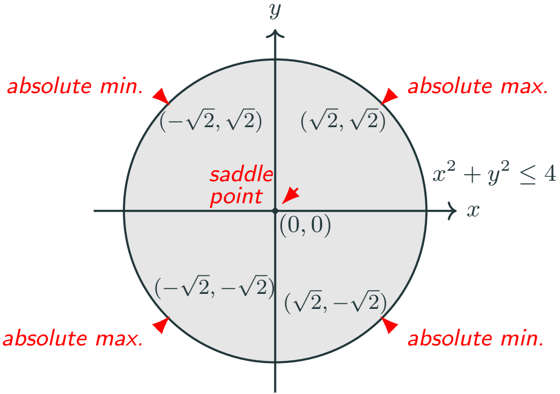

Extreme Values of Functions Defined on Restricted Domains

EXAMPLE

Find the extreme values of the function $f(x, y) = x^2ye^{-(x+y)}$ on the triangular region T given by $x \geq 0$, $y \geq 0$, and $x + y \leq 4$.

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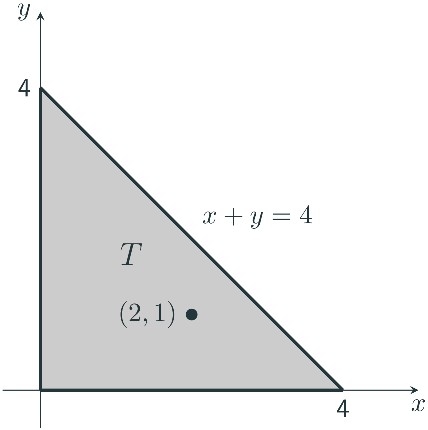

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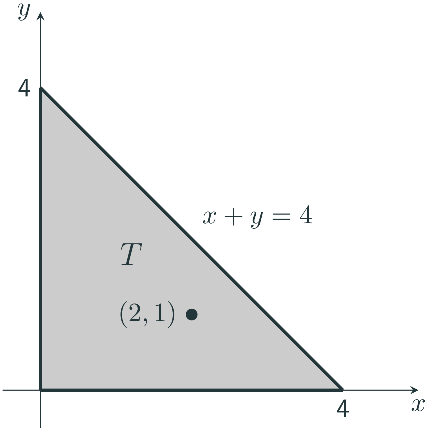

- **Finding critical points:**

$$0 = f_1(x, y) = xy(2 - x)e^{-(x+y)}$$

$$0 = f_2(x, y) = x^2(1 - y)e^{-(x+y)}$$

$$\Longleftrightarrow x = 0,\ y = 0, \text{ or } x = 2,$$

$$\Longleftrightarrow x = 0 \text{ or } y = 1.$$

The critical points are $(0, y)$ for any y and $(2, 1)$. Only $(2, 1)$ is an interior point of T.
$f(2, 1) = \frac{4}{e^3} \approx 0.199$.

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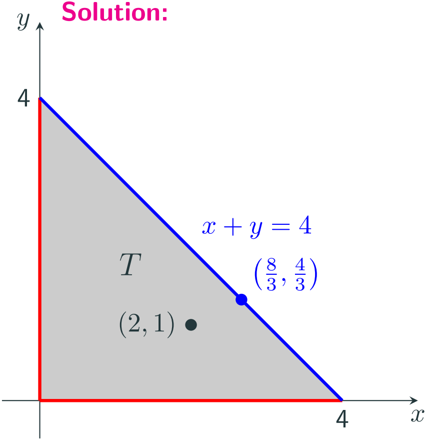

- **Boundary points:**

  $x = 0$ and $y = 0 \quad 0 \leq x, y \leq 4$

  $y = 4 - x, \quad 0 \leq x \leq 4$

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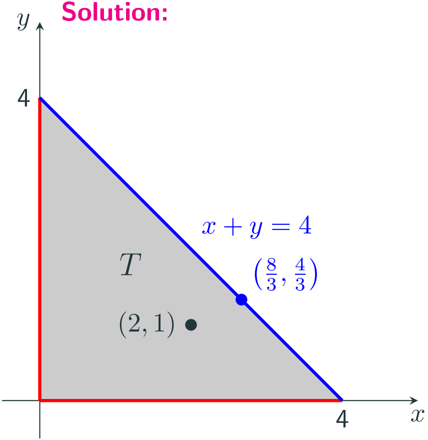

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  For $0 \leq x \leq 4$,

  $$g(x) = f(x, 4-x) = x^2(4-x)e^{-4},$$

  $g(0) = g(4) = 0$ and $g(x) > 0$ if $x \neq 0, 4$

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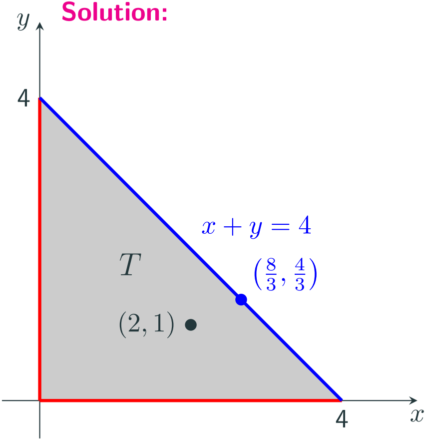

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$\Rightarrow x = 0$ and $x = 8/3$.

$$g\left(\frac{8}{3}\right) = f\left(\frac{8}{3}, \frac{4}{3}\right)$$

$$= \frac{256}{27}e^{-4} \approx 0.174 < f(2, 1).$$

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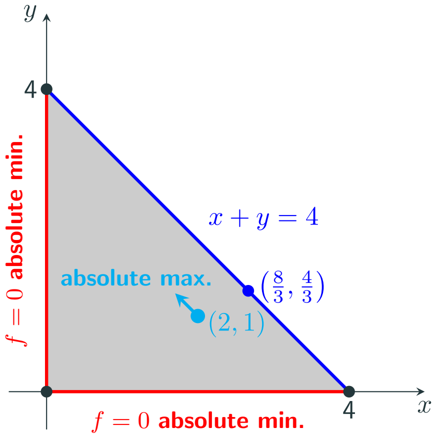

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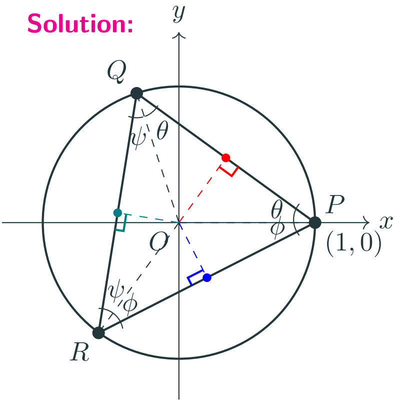

$$2\theta + 2\phi + 2\psi = \pi$$

- Without loss of generality, we may assume the circle is $x^2 + y^2 = 1$ and one vertex at $(0, 1)$.

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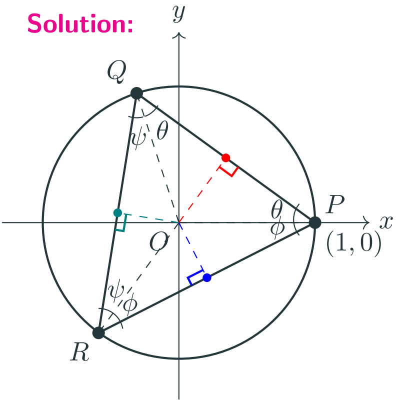

$$2\theta + 2\phi + 2\psi = \pi$$

- Without loss of generality, we may assume the circle is $x^2 + y^2 = 1$ and one vertex at $(0, 1)$.
- Try to draw as large triangle as possible (which should obviously contain the origin inside).

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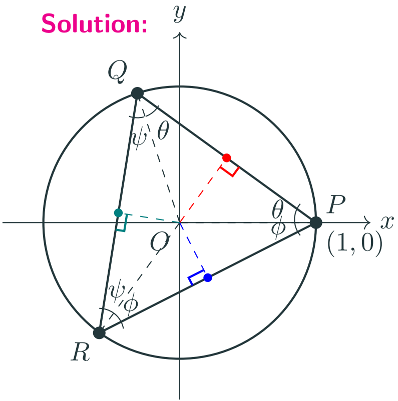

$$A = 2 \times \frac{1}{2}\sin\theta\cos\theta + 2 \times \frac{1}{2}\sin\phi\cos\phi + 2 \times \frac{1}{2}\sin\psi\cos\psi = \frac{1}{2}(\sin 2\theta + \sin 2\phi + \sin 2\psi)$$

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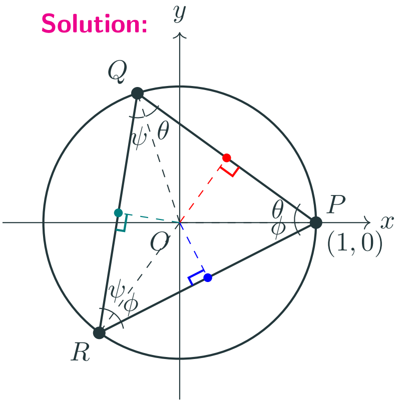

$2\theta + 2\phi + 2\psi = \pi$

$$A = 2 \times \frac{1}{2}\sin\theta\cos\theta + 2 \times \frac{1}{2}\sin\phi\cos\phi + 2 \times \frac{1}{2}\sin\psi\cos\psi$$

$$= \frac{1}{2}(\sin 2\theta + \sin 2\phi + \sin 2\psi)$$

Since $2\psi = \pi - 2(\theta + \phi)$, area can be written as

$$A(\theta, \phi) = \frac{1}{2}(\sin 2\theta + \sin 2\phi + \sin 2(\theta + \phi))$$

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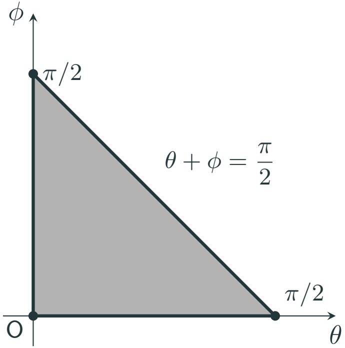
Solution: $A = A(\theta, \phi) = \frac{1}{2}(\sin 2\theta + \sin 2\phi + \sin 2(\theta + \phi))$

The domain of A is the triangle $\theta \geq 0, \phi \geq 0, \theta + \phi \leq \pi/2$. $A = 0$ at the vertices of the triangle and is positive elsewhere.



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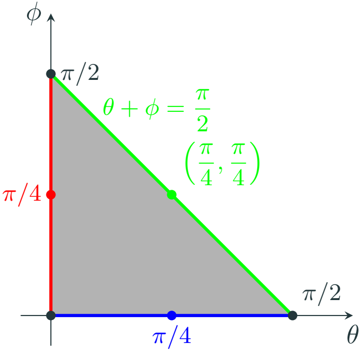

On the edge $\theta = 0$ we have

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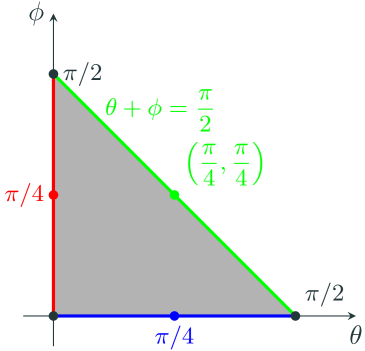

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Similarly, on $\phi = 0, A(\theta, 0) \leq 1 = A(\pi/4, 0)$.

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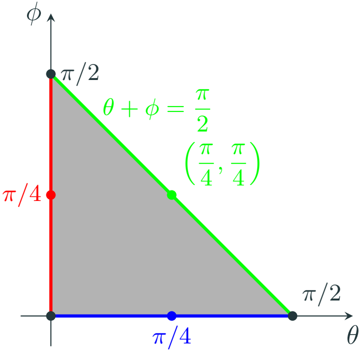

On the edge $\theta + \phi = \pi/2$ we have

$$A\left(\theta, \frac{\pi}{2} - \theta\right) = \frac{1}{2}(\sin 2\theta + \sin(\pi - 2\theta)) = \sin 2\theta \le 1 = A\left(\frac{\pi}{4}, \frac{\pi}{4}\right).$$

EXAMPLE

Among all triangles with vertices on a given circle, find those that have the largest area.

Solution: We must now check for any interior critical points of $A(\theta, \phi)$.

$$0 = \frac{\partial A}{\partial \theta} = \cos 2\theta + \cos(2\theta + 2\phi),$$
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$$\cos 2\theta + \cos 4\theta = 0 \Rightarrow 2\cos^2 2\theta + \cos 2\theta - 1 = 0$$
$$\Rightarrow (2\cos 2\theta - 1)(\cos 2\theta + 1) = 0$$
$$\cos 2\theta = \frac{1}{2} \quad \text{or} \quad \cos 2\theta = -1 \Rightarrow \theta = \phi = \pi/6.$$

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Among all triangles with vertices on a given circle, find those that have the largest area.

Solution: Finally,

$$A\left(\frac{\pi}{6},\frac{\pi}{6}\right)=\frac{1}{2}\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right)=\frac{3\sqrt{3}}{4}>1,$$

which is the absolute maximum area.

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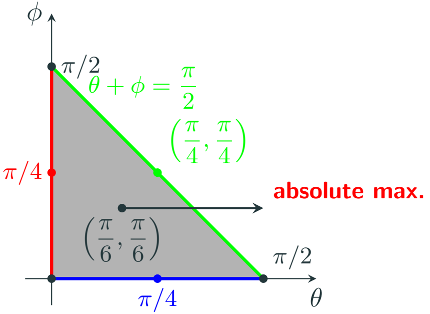
$$A\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4} > 1,$$

which is the absolute maximum area. The largest triangle is equilateral.



Linear Programming

Linear programming is a branch of linear algebra that develops systematic techniques for finding maximum or minimum values of a linear function subject to several linear inequality constraints.

The inequality $ax + by \leq c$ is an example of a linear inequality in two variables. The *solution set* of this inequality consists of a half-plane lying on one side of the straight line $ax + by = c$.

Linear Programming - Example

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Find the maximum value of $F(x, y) = 2x + 7y$ subject to the constraints $x + 2y \leq 6, \quad 2x + y \leq 6, \quad x \geq 0,$ and $y \geq 0.$

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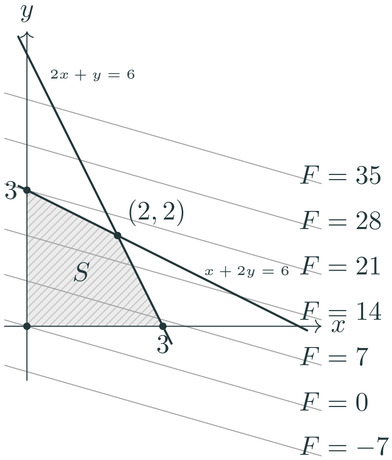

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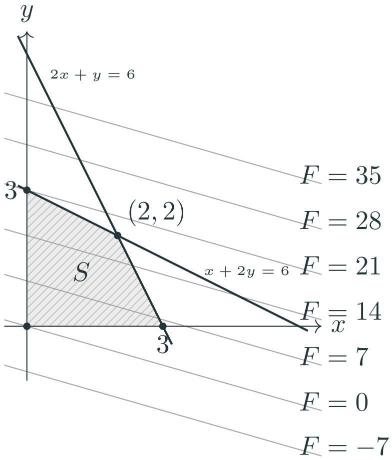

PROBLEM:

What is the greatest number c for which the level curve $F(x, y) = c$ intersects the shaded region?

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