MAT124 MATHEMATICS II

Double Integrals, Improper Integrals and a Mean-Value Theorem, Double Integrals in Polar Coordinates, Change of Variables in Double Integrals

Outline

Double Integrals

- Iteration of Double Integrals in Cartesian Coordinates
- Improper Integrals
- A Mean-Value Theorem for Double Integrals
- Double Integrals in Polar Coordinates
- Change of Variables in Double Integrals

Double Integrals

Iteration of Double Integrals in Cartesian Coordinates

Example

EXAMPLE

Evaluate the iterated integral $I = \int_0^1 dx \int_{\sqrt{x}}^1 e^{y^3}\, dy.$

Iteration of Double Integrals in Cartesian Coordinates

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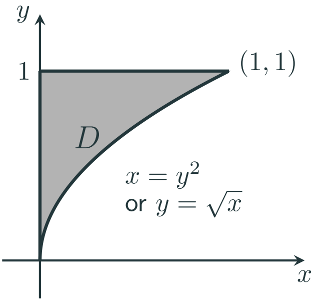
Evaluate the iterated integral $I = \int_0^1 dx \int_{\sqrt{x}}^1 e^{y^3}\, dy$.

Solution:

$$I = \iint_D e^{y^3}\, dA.$$



$$\begin{aligned} I &= \int_0^1 dy \int_0^{y^2} e^{y^3}\, dx \\ &= \int_0^1 e^{y^3} dy \int_0^{y^2} dx \\ &= \int_0^1 y^2 e^{y^3}\, dy = \left. \frac{e^{y^3}}{3} \right|_0^1 = \frac{e-1}{3}. \end{aligned}$$

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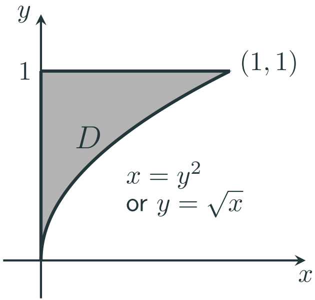

$$I = \iint\limits_D e^{y^3}\, dA.$$

Note that other order of iterated integral

$$I = \int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dy\, dx$$

can't integrate simply.

Improper Integrals of Positive Functions

An improper integral of a function f satisfying $f(x, y) \geq 0$ on the domain D must either exist (i.e., converge to a finite value) or be infinite (diverge to infinity).

Improper Integrals of Positive Functions

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Evaluate $I = \iint_R e^{-x^2}\, dA$. Here, R is the region where $x \geq 0$ and $-x \leq y \leq x$.

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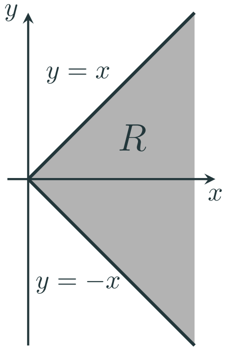

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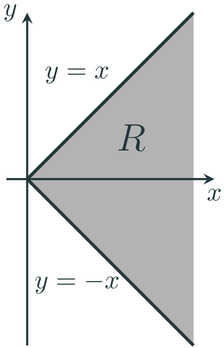

$$
\begin{aligned}
I &= \int_0^\infty dx \int_{-x}^{x} e^{-x^2}\,dy \\
&= \int_0^\infty e^{-x^2} dx \int_{-x}^{x} dy \\
&= 2\int_0^\infty x e^{-x^2}\,dx.
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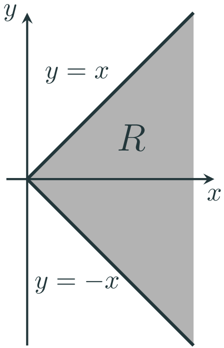

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$$\begin{aligned} I &= 2 \lim_{r\to\infty} \int_0^r x e^{-x^2}\,dx \\ &= 2 \lim_{r\to\infty} \left(-\frac{1}{2} e^{-x^2}\right)\Bigg|_0^r \\ &= \lim_{r\to\infty} \left(1 - e^{-r^2}\right) = 1. \end{aligned}$$

The given integral converges; its value is 1.

Improper Integrals of Positive Functions

Example

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If D is the region lying above the x-axis, under the curve $y = 1/x$, and to the right of the line $x = 1$, determine whether the double integral $\displaystyle\iint\limits_D \frac{dA}{x+y}$ converges or diverges.

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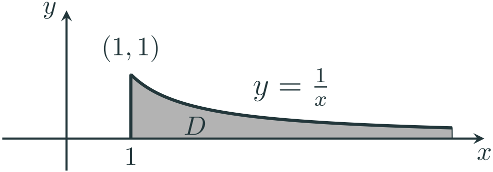

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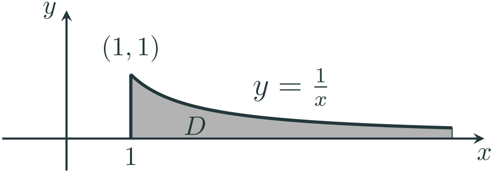

$$\iint_D \frac{dA}{x+y} = \int_1^\infty dx \int_0^{1/x} \frac{dy}{x+y} = \int_1^\infty \ln(x+y)\Big|_{y=0}^{y=1/x} dx$$

$$= \int_1^\infty \left(\ln\left(x + \frac{1}{x}\right) - \ln x\right) dx = \int_1^\infty \ln\left(\frac{x+\frac{1}{x}}{x}\right) dx$$

$$= \int_1^\infty \ln\left(1 + \frac{1}{x^2}\right) dx.$$

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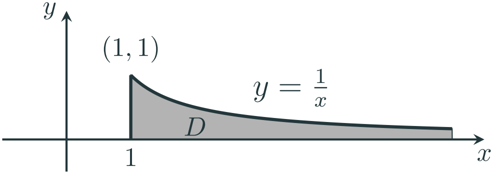

Since $0 < \ln(1+u) < u$ if $u > 0$, we have

$$0 < \iint_D \frac{dA}{x+y} < \int_1^\infty \frac{1}{x^2}dx = 1.$$

Therefore, the given integral converges, and its value lies between 0 and 1.

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Evaluate $\displaystyle\iint_D \frac{1}{(x+y)^2}\,dA$, where D is the region $0 \le x \le 1$, $0 \le y \le x^2$.

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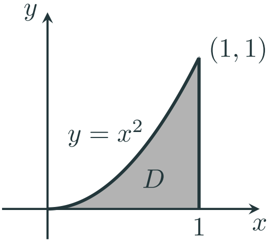

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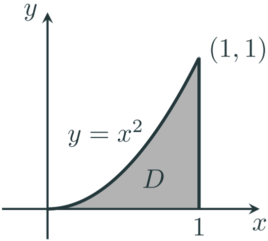

$$\begin{aligned}\iint_D \frac{1}{(x+y)^2}\,dA &= \lim_{c\to 0+}\int_c^1 dx \int_0^{x^2} \frac{1}{(x+y)^2}\,dy \\ &= \lim_{c\to 0+}\int_c^1 dx \left(-\frac{1}{x+y}\right)\Bigg|_{y=0}^{y=x^2} \\ &= \lim_{c\to 0+}\int_c^1 \left(\frac{1}{x} - \frac{1}{x^2+x}\right) dx = \int_0^1 \frac{1}{x+1}\,dx \\ &= \ln(x+1)\Big|_0^1 = \ln 2.\end{aligned}$$

A Mean-Value Theorem for Double Integrals

A set D in the plane is said to be connected if any two points in it can be joined by a continuous parametric curve $x = x(t)$, $y = y(t)$, $(0 \leq t \leq 1)$, lying in D.

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If the function $f(x, y)$ is continuous on a closed, bounded, connected set D in the xy-plane, then there exists a point (x_0, y_0) in D such that

$$\iint_D f(x, y)\, dA = f(x_0, y_0) \times (\text{area of } D).$$

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The average value or mean value of an integrable function $f(x, y)$ over the set D is the number

$$\bar{f} = \frac{1}{\text{area of } D} \iint\limits_D f(x, y)\, dA.$$

A Mean-Value Theorem for Double Integrals

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A large number of points (x, y) are chosen at random in the triangle T with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$. What is the approximate average value of $x^2 + y^2$ for these points?

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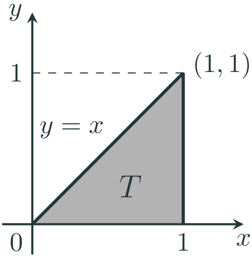

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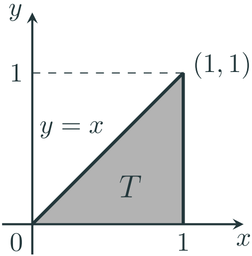

$$\frac{1}{1/2}\iint_T (x^2+y^2)\,dA$$

$$= 2\int_0^1 dx \int_0^x (x^2+y^2)\,dy$$

$$= 2\int_0^1 \left(x^2y + \frac{1}{3}y^3\right)\Bigg|_{y=0}^{y=x} dx$$

$$= \frac{8}{3}\int_0^1 x^3 dx = \frac{2}{3}.$$

Double Integrals in Polar Coordinates

Consider the problem of finding the volume V of the solid region lying above the xy-plane and beneath the paraboloid $z = 1 - x^2 - y^2$.

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$$V = \iint\limits_{x^2+y^2\le 1} (1 - x^2 - y^2)\, dA = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy.$$

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Evaluating this iterated integral would require considerable effort. However, we can express the same volume in terms of polar coordinates as

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$dA = ?$ (in polar coordinates)

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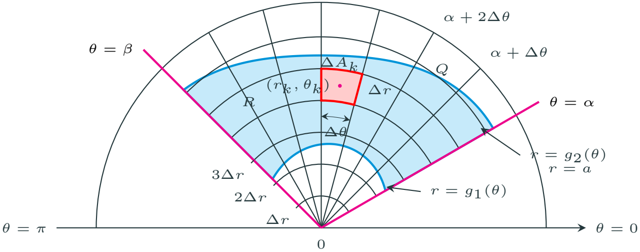
Suppose that a function $f(r, \theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \leq g_1(\theta) \leq g_2(\theta) \leq a$ for every value of θ between α and β. Then R lies in a fan-shaped region Q defined by the inequalities $0 \leq r \leq a$ and $\alpha \leq \theta \leq \beta$.



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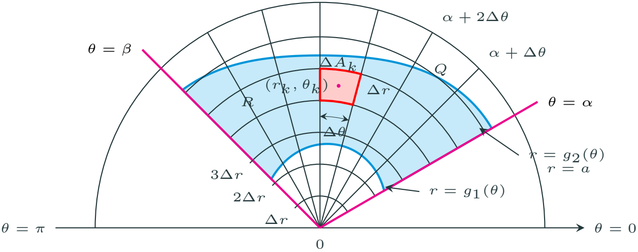
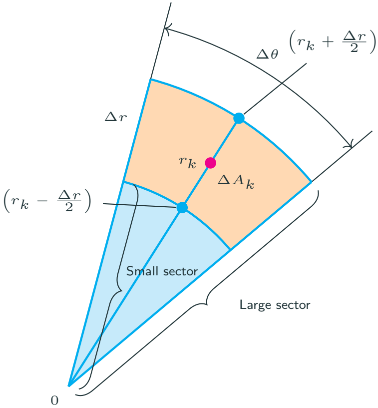


$$S_n=\sum_{k=1}^{n} f(r_k,\theta_k)\Delta A_k \quad \xrightarrow{f \text{ is continuous}} \quad \lim_{n\to\infty} S_n=\iint_R f(r,\theta)\,dA.$$

Double Integrals in Polar Coordinates

Let's consider the small section:



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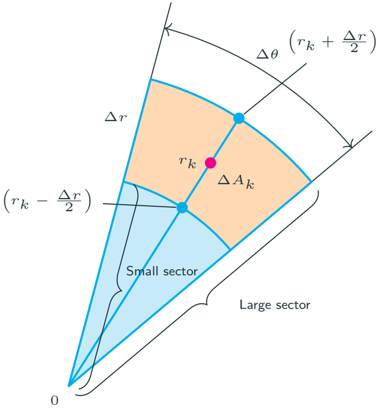

$$\Delta A_k = \begin{pmatrix} \text{area of} \\ \text{large sector} \end{pmatrix} - \begin{pmatrix} \text{area of} \\ \text{small sector} \end{pmatrix}$$

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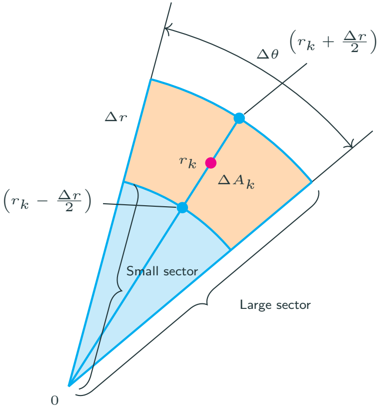
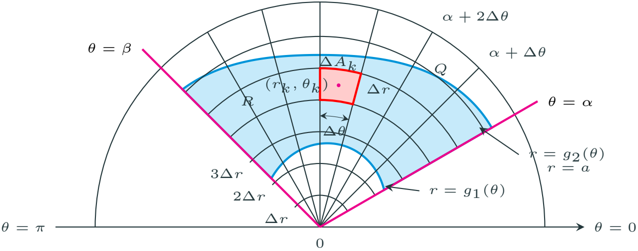


$$\Delta A_k = \begin{pmatrix}\text{area of}\\ \text{large sector}\end{pmatrix} - \begin{pmatrix}\text{area of}\\ \text{small sector}\end{pmatrix}$$

$$= \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right]$$

$$= \frac{\Delta\theta}{2}(2r_k\Delta r) = r_k\Delta r\Delta\theta.$$

Double Integrals in Polar Coordinates



Thus,

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k = \sum_{k=1}^{n} f(r_k, \theta_k) r_k \Delta r \Delta\theta$$

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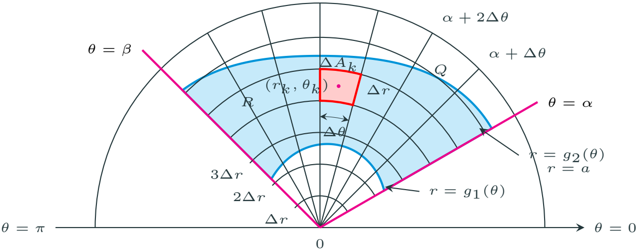

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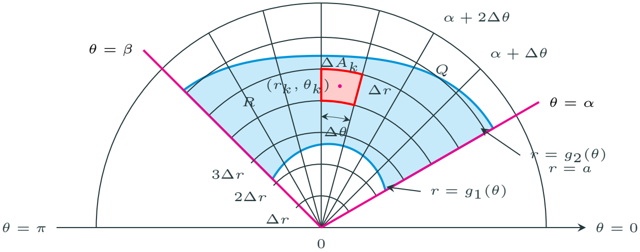

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Finally:

$$\iint_R f(r,\theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Double Integrals in Polar Coordinates

Example

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If R is that part of the annulus $0 < a^2 \le x^2 + y^2 \le b^2$ lying in the first quadrant and below the line $y = x$, evaluate $I = \displaystyle\iint\limits_R \frac{y^2}{x^2}\, dA$.

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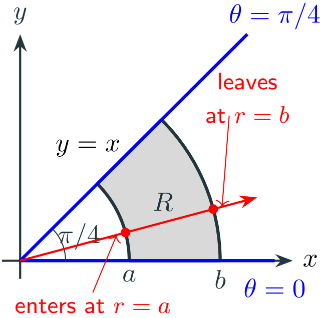

$$\frac{y^2}{x^2} = \frac{r^2 \sin^2\theta}{r^2\cos^2\theta} = \tan^2\theta.$$

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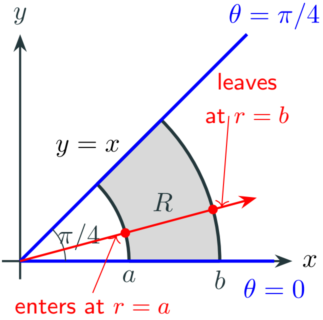

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$$\begin{aligned} I &= \int_{\theta=0}^{\theta=\pi/4} \int_{r=a}^{r=b} r\tan^2\theta\, dr\, d\theta \\ &= \int_0^{\pi/4} \tan^2\theta \left(\frac{r^2}{2}\Big|_a^b \right) d\theta \\ &= \frac{b^2-a^2}{2} \int_0^{\pi/4} \underbrace{\tan^2\theta}_{\sec^2\theta - 1}\, d\theta \\ &= \frac{b^2-a^2}{2}\left(1 - \frac{\pi}{4}\right). \end{aligned}$$

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Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

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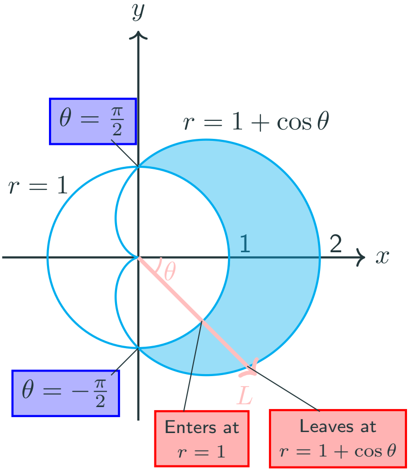

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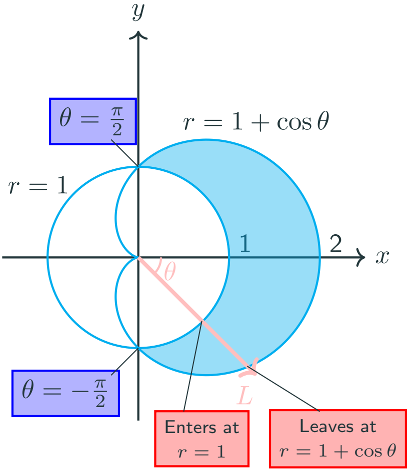

$$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} f(r,\theta)\, r \, dr \, d\theta$$

Double Integrals in Polar Coordinates

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\,dr\,d\theta.$$

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Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

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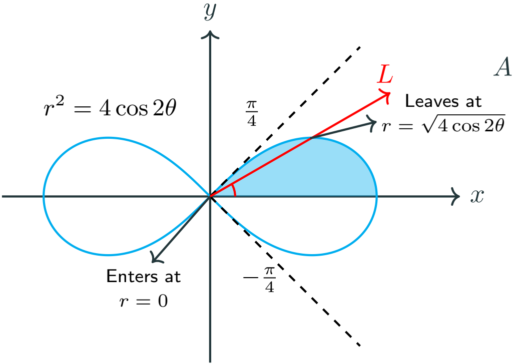

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$$

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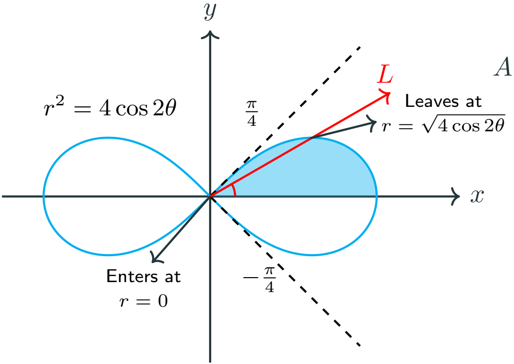
Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Solution:



$$
\begin{aligned}
A &= 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta \\
&= 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta \\
&= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta \\
&= 4\left[\sin 2\theta\right]_0^{\pi/4} \\
&= 4.
\end{aligned}
$$

Double Integrals in Polar Coordinates

Example

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Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$.

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$$0 \le x \le 1$$

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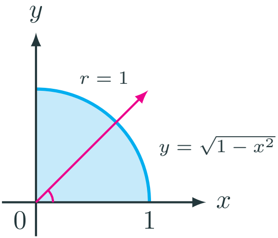

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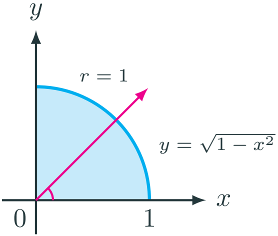

$$\begin{aligned} &\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx \\ &= \int_0^{\pi/2} \int_0^1 (r^2)\,r\,dr\,d\theta \\ &= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta \\ &= \int_0^{\pi/2} \frac{1}{4}\,d\theta = \frac{\pi}{8}. \end{aligned}$$

Double Integrals in Polar Coordinates

Example

EXAMPLE

Find the volume of the solid region lying inside both the sphere $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + y^2 = 2ay$, where $a > 0$.

Double Integrals in Polar Coordinates

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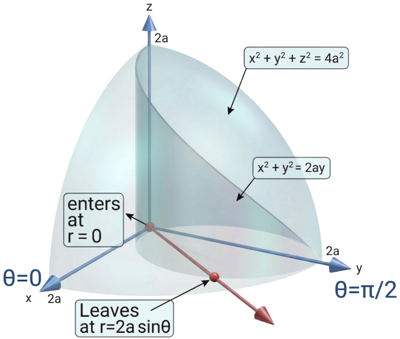
Solution: The sphere is centred at the origin and has radius $2a$. The equation of the cylinder becomes $x^2 + (y - a)^2 = a^2$.



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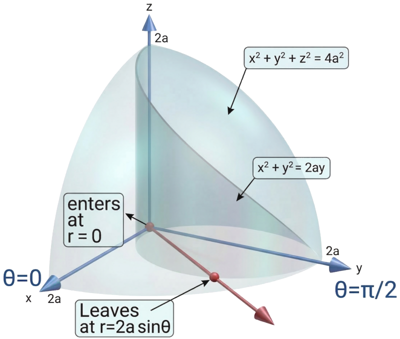

$$V = 4\int_0^{\pi/2}\int_0^{2a\sin\theta} \sqrt{4a^2 - r^2}\, r\, dr\, d\theta$$

$\vdots$ (Exercise!)

$$= \frac{16}{9}(3\pi - 4)a^3 \text{ cubic units.}$$

Change of Variables in Double Integrals

Suppose that x and y are expressed as functions of two other variables u and v by the equations

$$x = x(u, v) \qquad \text{and} \qquad y = y(u, v).$$

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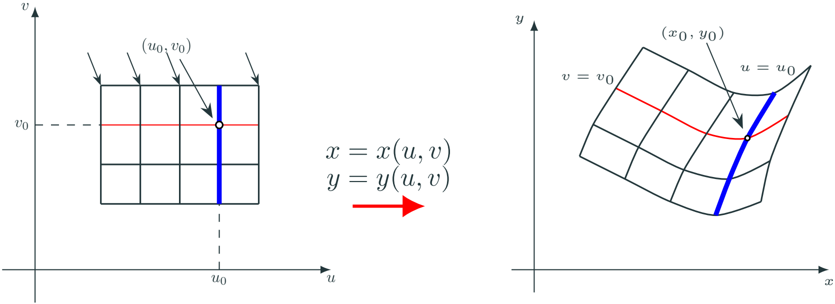
$$x = x(u, v) \qquad \text{and} \qquad y = y(u, v).$$

We regard these equations as defining a **transformation** (or mapping) from points (u, v) in a uv-Cartesian plane to points (x, y) in the xy-plane.



Change of Variables in Double Integrals

We say that the transformation is one-to-one from the set S in the uv-plane *onto* the set D in the xy-plane provided:

(i) every point in S gets mapped to a point in D,
(ii) every point in D is the image of a point in S, and
(iii) different points in S get mapped to different points in D.

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(iii) different points in S get mapped to different points in D.

If the transformation is one-to-one, the defining equations can be solved for u and v as functions of x and y, and the resulting inverse transformation,

$$u = u(x, y)$$

$$v = v(x, y),$$

is one-to-one from D onto S.

Change of Variables in Double Integrals

Suppose that the functions $x = x(u, v)$, $y = y(u, v)$ have continuous first partial derivatives and that

$$\frac{\partial(x, y)}{\partial(u, v)} \neq 0 \text{ at } (u, v).$$

Change of Variables in Double Integrals

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Then the transformation

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$$

is one-to-one near (u, v) and the inverse transformation has continuous first partial derivatives. We also have

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{\frac{\partial(x, y)}{\partial(u, v)}}.$$

Change of Variables in Double Integrals

A one-to-one transformation can be used to transform the double integral

$$\iint\limits_D f(x,y)\,dA$$

to a double integral over the corresponding set S in the uv-plane.

$$\iint\limits_D f(x,y)\,\underbrace{dA}_{dx\,dy} \qquad = \qquad \iint\limits_S f(x(u,v),y(u,v))\,dA$$

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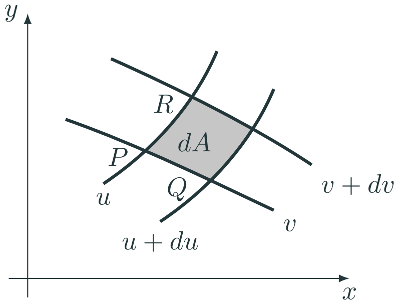
$dA = ?$ (in terms of $du\,dv$)

Change of Variables in Double Integrals



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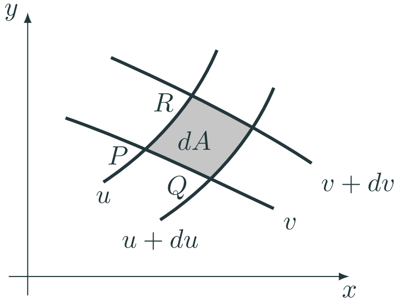

$$dA \approx \left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

$$\overrightarrow{PQ} = dx\,\mathbf{i} + dy\,\mathbf{j}$$

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$

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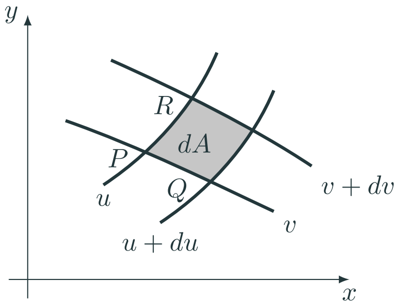
Change of Variables in Double Integrals



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Since $dv = 0$ along v curve,

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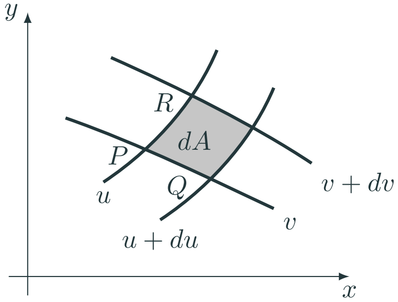
Change of Variables in Double Integrals



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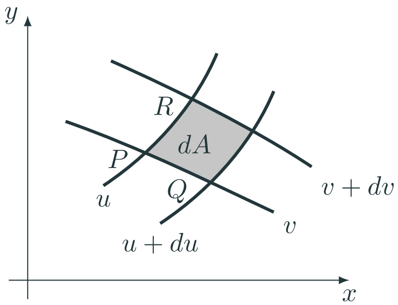
Similarly,

$$\boxed{\overrightarrow{PR} = \frac{\partial x}{\partial v}dv\,\mathbf{i} + \frac{\partial y}{\partial v}dv\,\mathbf{j}}$$

Change of Variables in Double Integrals



$$dA = \left| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} du & \frac{\partial y}{\partial u} du & 0 \\ \frac{\partial x}{\partial v} dv & \frac{\partial y}{\partial v} dv & 0 \end{matrix} \right|$$

$$= \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Change of Variables in Double Integrals

Theorem - Change of variables formula for double integrals

Let $x = x(u, v)$, $y = y(u, v)$ be a one-to-one transformation from a domain S in the uv-plane onto a domain D in the xy-plane. Suppose that the functions x and y, and their first partial derivatives with respect to u and v, are continuous in S. If $f(x, y)$ is integrable on D, and if $g(u, v) = f(x(u, v), y(u, v))$, then g is integrable on S and

$$\iint_D f(x, y)\, dx\, dy = \iint_S g(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv.$$