

# **MAT124 MATHEMATICS II**

Change of Variables in Double Integrals, Triple Integrals

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Double Integrals

Change of Variables in Double Integrals

Triple Integrals

# Double Integrals

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## Change of Variables in Double Integrals

Suppose that  $x$  and  $y$  are expressed as functions of two other variables  $u$  and  $v$  by the equations

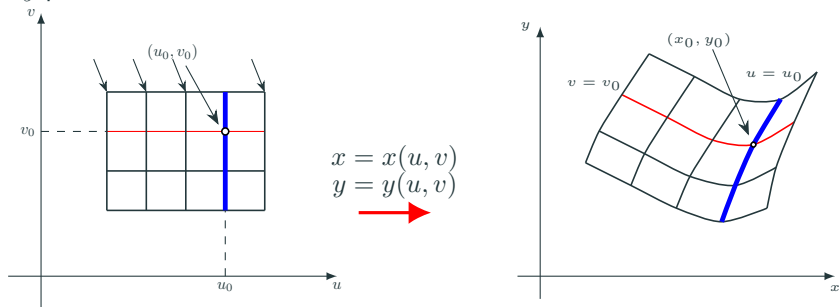
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# Change of Variables in Double Integrals

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$$x = x(u, v) \quad \text{and} \quad y = y(u, v).$$

We regard these equations as defining a **transformation** (or mapping) from points  $(u, v)$  in a  $uv$ -Cartesian plane to points  $(x, y)$  in the  $xy$ -plane.



## Change of Variables in Double Integrals

We say that the transformation is **one-to-one** from the set  $S$  in the  $uv$ -plane *onto* the set  $D$  in the  $xy$ -plane provided:

- (i) every point in  $S$  gets mapped to a point in  $D$ ,
- (ii) every point in  $D$  is the image of a point in  $S$ , and
- (iii) different points in  $S$  get mapped to different points in  $D$ .

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If the transformation is one-to-one, the defining equations can be solved for  $u$  and  $v$  as functions of  $x$  and  $y$ , and the resulting **inverse transformation**,

$$u = u(x, y)$$

$$v = v(x, y),$$

is one-to-one from  $D$  onto  $S$ .

## Change of Variables in Double Integrals

Suppose that the functions  $x = x(u, v)$ ,  $y = y(u, v)$  have continuous first partial derivatives and that

$$\frac{\partial(x, y)}{\partial(u, v)} \neq 0 \text{ at } (u, v).$$

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Then the transformation

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$$

is one-to-one near  $(u, v)$  and the inverse transformation has continuous first partial derivatives. We also have

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{\frac{\partial(x, y)}{\partial(u, v)}}.$$

## Change of Variables in Double Integrals

A one-to-one transformation can be used to transform the double integral

$$\iint_D f(x, y) dA$$

to a double integral over the corresponding set  $S$  in the  $uv$ -plane.

$$\iint_D f(x, y) \underbrace{dA}_{dx dy} = \iint_S f(x(u, v), y(u, v)) dA$$

## Change of Variables in Double Integrals

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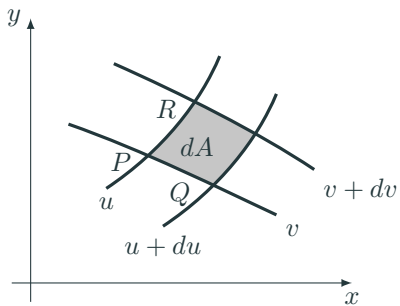
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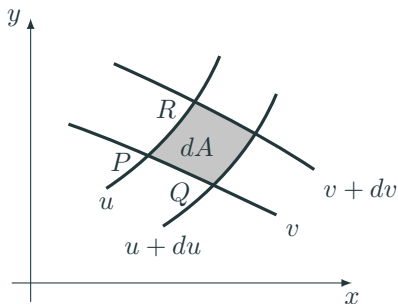
$$dA = ? \text{ (in terms of } du dv \text{)}$$

# Change of Variables in Double Integrals



# Change of Variables in Double Integrals

$$dA \approx \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right|$$

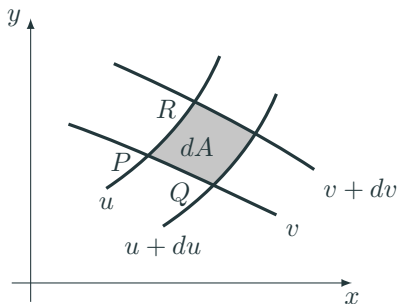


$$\overrightarrow{PQ} = dx \mathbf{i} + dy \mathbf{j}$$

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

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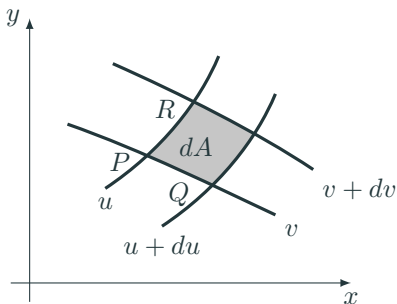
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Since  $dv = 0$  along  $v$  curve,

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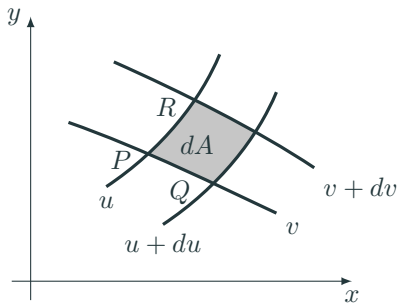
Since  $dv = 0$  along  $v$  curve,

$$\overrightarrow{PQ} = \frac{\partial x}{\partial u} du \mathbf{i} + \frac{\partial y}{\partial u} du \mathbf{j}$$

Similarly,

$$\overrightarrow{PR} = \frac{\partial x}{\partial v} dv \mathbf{i} + \frac{\partial y}{\partial v} dv \mathbf{j}$$

# Change of Variables in Double Integrals



$$\begin{aligned} dA &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} du & \frac{\partial y}{\partial u} du & 0 \\ \frac{\partial x}{\partial v} dv & \frac{\partial y}{\partial v} dv & 0 \end{vmatrix} \\ &= \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \end{aligned}$$

## Change of Variables in Double Integrals

### Theorem - Change of variables formula for double integrals

Let  $x = x(u, v)$ ,  $y = y(u, v)$  be a one-to-one transformation from a domain  $S$  in the  $uv$ -plane onto a domain  $D$  in the  $xy$ -plane. Suppose that the functions  $x$  and  $y$ , and their first partial derivatives with respect to  $u$  and  $v$ , are continuous in  $S$ . If  $f(x, y)$  is integrable on  $D$ , and if  $g(u, v) = f(x(u, v), y(u, v))$ , then  $g$  is integrable on  $S$  and

$$\iint_D f(x, y) dx dy = \iint_S g(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.$$

# Change of Variables in Double Integrals

## Example

### EXAMPLE

Use an appropriate change of variables to find the area of the elliptic disk

E given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1.$$

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**Solution:** Under the transformation  $x = au$ ,  $y = bv$ , the elliptic disk  $E$  is the one-to-one image of the circular disk  $D$  given by  $u^2 + v^2 \leq 1$ . Assuming  $a > 0$  and  $b > 0$ , we have

$$dx \, dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \, dv = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} du \, dv = ab \, du \, dv.$$

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Therefore, the area of  $E$  is given by

$$\iint_E 1 \, dx \, dy = \iint_D ab \, du \, dv = ab \times (\text{area of } D) = \pi ab \text{ square units.}$$

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Find the area of the finite plane region bounded by the four parabolas

$$y = x^2, \quad y = 2x^2, \quad x = y^2, \quad \text{and} \quad x = 3y^2.$$

# Change of Variables in Double Integrals

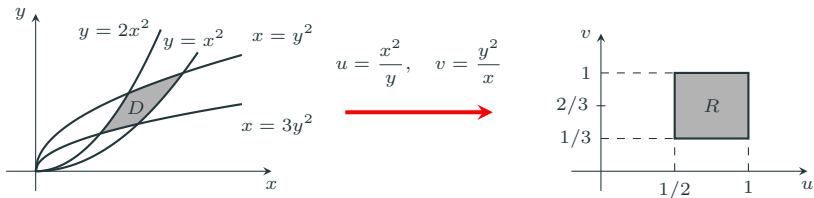
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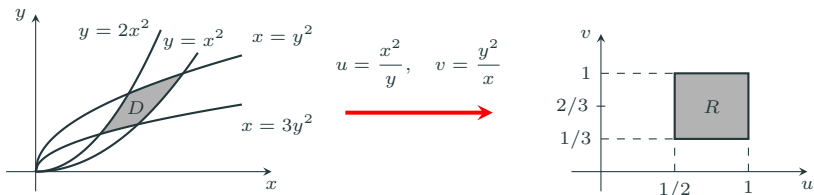
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**Solution:** Let  $u = \frac{x^2}{y}$  and  $v = \frac{y^2}{x}$ .



Then the region  $D$  corresponds to the rectangle  $R$  in the  $uv$ -plane given by  $\frac{1}{2} \leq u \leq 1$  and  $\frac{1}{3} \leq v \leq 1$ .

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = 4 - 1 = 3 \quad \Rightarrow \quad \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{1}{3}$$

# Change of Variables in Double Integrals

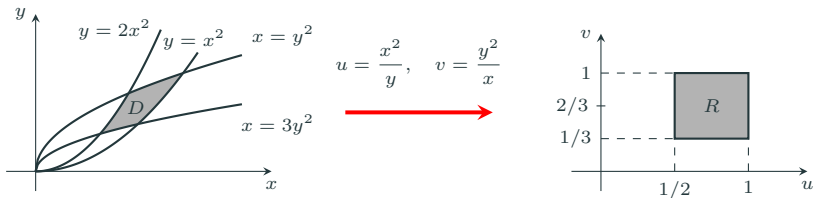
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Finally,

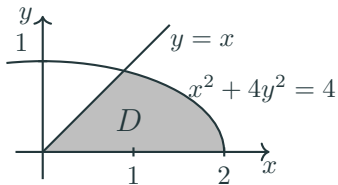
$$\iint_D dx dy = \iint_R \frac{1}{3} du dv = \frac{1}{3} \times \frac{1}{2} \times \frac{2}{3} = \frac{1}{9} \text{ square units.}$$

# Change of Variables in Double Integrals

## Example

### EXAMPLE

Evaluate  $I = \iint_D \frac{y}{x} dx dy$ , where  $D$  is the shaded region.

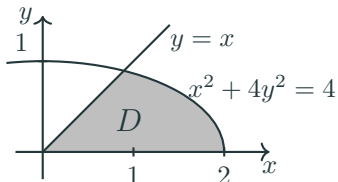


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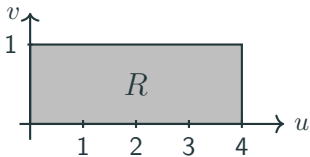
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Evaluate  $I = \iint_D \frac{y}{x} dx dy$ , where  $D$  is the shaded region.



**Solution:** Let  $u = x^2 + 4y^2$ ,  $v = \frac{y}{x}$ , then the region  $D$  will be the region  $R$ :

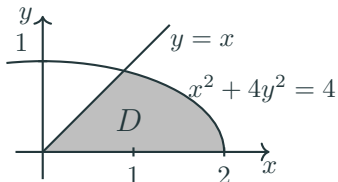


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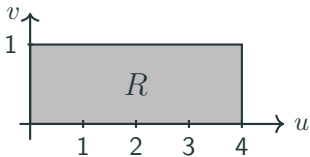
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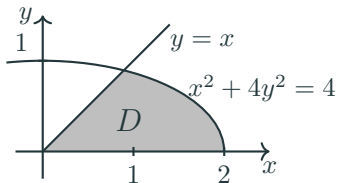
$$\begin{aligned} \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} 2x & 8y \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = 2 + 8\frac{y^2}{x^2} \\ &= 2 + 8v^2 \Rightarrow \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{2 + 8v^2} \end{aligned}$$

# Change of Variables in Double Integrals

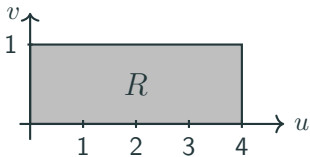
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$$I = \iint_R \frac{v}{2 + 8v^2} du dv = \int_0^4 du \int_0^1 \frac{v}{2 + 8v^2} dv$$

$$\text{By } w = 2 + 8v^2, \quad dw = 16v dv$$

$$= \frac{4}{16} \int_2^{10} \frac{dw}{w} = \frac{1}{4} (\ln 10 - \ln 2) = \frac{1}{4} \ln 5.$$

# Change of Variables in Double Integrals

## Example

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Evaluate  $I = \iint_T (x + y)^3 dx dy$  over the triangle  $T$  with vertices  $(0, 0)$ ,  $(1, 1)$ , and  $(2, 0)$ .

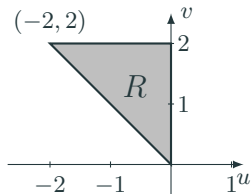
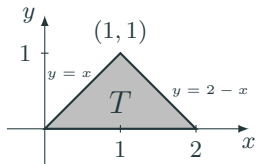
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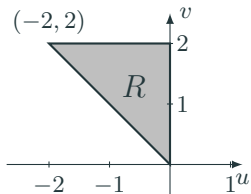
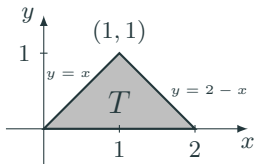
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$$dx dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \frac{1}{2} du dv$$

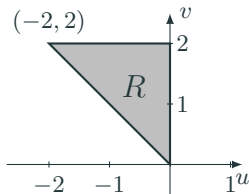
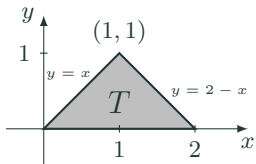
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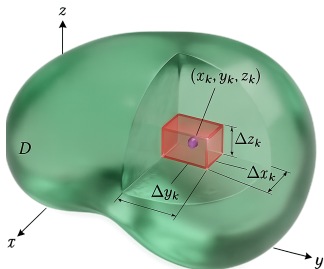
$$I = \frac{1}{2} \iint_R v^3 du dv = \frac{16}{5}$$

# Triple Integrals

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# Triple Integrals

$F(x, y, z)$  is a function defined on a closed, bounded region  $D$  in space. We partition a rectangular boxlike region containing  $D$  into rectangular cells by planes parallel to the coordinate axes.

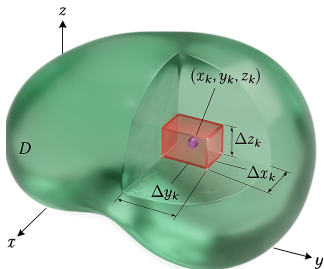


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We form the sum

$$S_n = \sum_{k=1}^n F(x_k, y_k, z_k) \Delta V_k.$$



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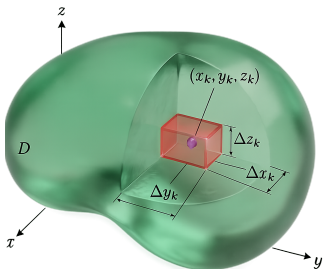
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As the norm of the partition approach zero, if a single limiting value is attained, no matter how the partitions and points are chosen, we say that  $F$  is integrable over  $D$ .

$$\lim_{n \rightarrow \infty} S_n = \iiint_D F(x, y, z) dV \quad \text{or}$$

$$\lim_{\|P\| \rightarrow 0} S_n = \iiint_D F(x, y, z) dx dy dz.$$



The norm of the partition  $\|P\| =$  the largest value among  $\Delta x_k, \Delta y_k, \Delta z_k$

# Triple Integrals

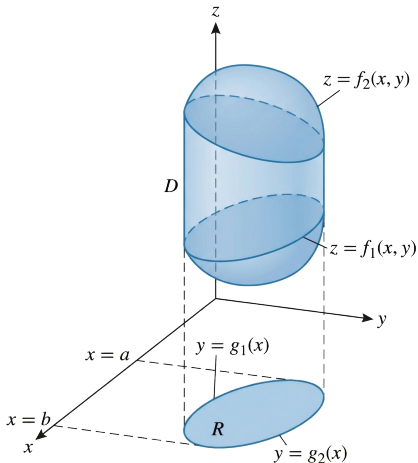
## Definition - Volume by triple integral

The **volume** of a closed, bounded region  $D$  in space is

$$V = \iiint_D dV.$$

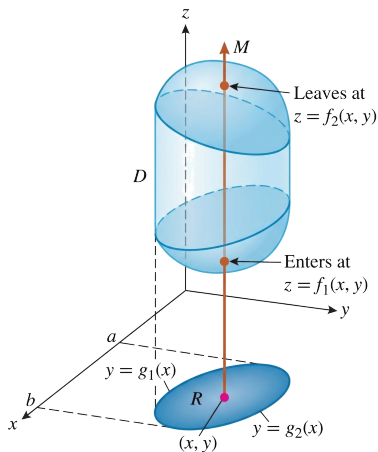
## Finding Limits of Integration in the Order $dz dy dx$

1. **Sketch:** Sketch the region  $D$  along with its “shadow”  $R$  (vertical projection) in the  $xy$ -plane. Label the upper and lower bounding surfaces of  $D$  and the upper and lower bounding curves of  $R$ .



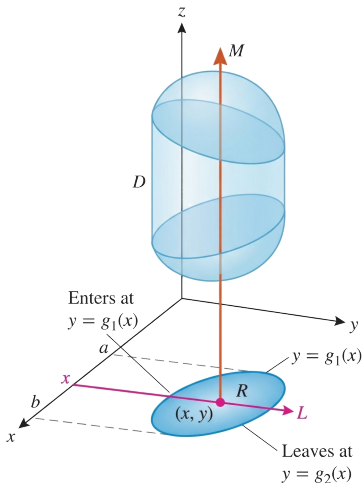
## Finding Limits of Integration in the Order $dz dy dx$

2. **Find the  $z$ -limits of integration:** Draw a line  $M$  passing through a typical point  $(x, y)$  in  $R$  parallel to the  $z$ -axis. As  $z$  increases,  $M$  enters  $D$  at  $z = f_1(x, y)$  and leaves at  $z = f_2(x, y)$ . These are the  $z$ -limits of integration.



## Finding Limits of Integration in the Order $dz dy dx$

3. **Find the  $y$ -limits of integration:** Draw a line  $L$  through  $(x, y)$  parallel to the  $y$ -axis. As  $y$  increases,  $L$  enters  $R$  at  $y = g_1(x)$  and leaves at  $y = g_2(x)$ . These are the  $y$ -limits of integration.



## Finding Limits of Integration in the Order $dz dy dx$

4. **Find the  $x$ -limits of integration:** Choose  $x$ -limits that include all lines through  $R$  parallel to the  $y$ -axis ( $x = a$  and  $x = b$  in the preceding figure). These are the  $x$ -limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z) dz dy dx.$$

# Triple Integrals

## Example

### EXAMPLE

Find the volume of the region  $D$  enclosed by the surfaces  $z = x^2 + 3y^2$  and  $z = 8 - x^2 - y^2$ .





# Change of Variables in Double Integrals

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Find the volume of the region  $D$  enclosed by the surfaces  $z = x^2 + 3y^2$  and  $z = 8 - x^2 - y^2$ .

**Solution:**

$$\begin{aligned} & \int_{-2}^2 \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz \, dy \, dx \\ &= \int_{-2}^2 \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2) \, dy \, dx \\ &= \int_{-2}^2 \left[ (8 - 2x^2)y - \frac{4}{3}y^3 \right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx \\ &= \int_{-2}^2 \left( 2(8 - 2x^2) \sqrt{\frac{4-x^2}{2}} - \frac{8}{3} \left( \frac{4-x^2}{2} \right)^{3/2} \right) dx \\ &= \int_{-2}^2 \left[ 8 \left( \frac{4-x^2}{2} \right)^{3/2} - \frac{8}{3} \left( \frac{4-x^2}{2} \right)^{3/2} \right] dx \\ &= \frac{4\sqrt{2}}{3} \int_{-2}^2 (4-x^2)^{3/2} dx = 8\pi\sqrt{2} \end{aligned}$$

# Change of Variables in Double Integrals

## Example

### EXAMPLE

Set up the limits of integration for evaluating the triple integral of a function  $F(x, y, z)$  over the tetrahedron  $D$  with vertices  $(0, 0, 0)$ ,  $(1, 1, 0)$ ,  $(0, 1, 0)$ , and  $(0, 1, 1)$ . Use the order of integration  $dy dz dx$ . Integrate  $F(x, y, z) = 1$  over the tetrahedron  $D$  in the order  $dz dy dx$ , and then integrate in the order  $dy dz dx$ .

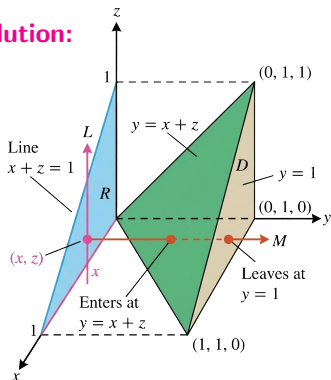
# Change of Variables in Double Integrals

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### Solution:









# Change of Variables in Double Integrals

## Example

### Solution:

$$\begin{aligned}V &= \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy dz dx \\&= \int_0^1 \int_0^{1-x} (1-x-z) dz dx \\&= \int_0^1 \left[ (1-x)z - \frac{1}{2}z^2 \right]_{z=0}^{z=1-x} dx \\&= \int_0^1 \left[ (1-x)^2 - \frac{1}{2}(1-x)^2 \right] dx \\&= \frac{1}{2} \int_0^1 (1-x)^2 dx \\&= \left[ -\frac{1}{6}(1-x)^3 \right]_0^1 \\&= \frac{1}{6}.\end{aligned}$$

$$\begin{aligned}V &= \int_0^1 \int_x^1 \int_0^{y-x} dz dy dx \\&= \int_0^1 \int_x^1 (y-x) dy dx \\&= \int_0^1 \left[ \frac{1}{2}y^2 - xy \right]_{y=x}^{y=1} dx \\&= \int_0^1 \left( \frac{1}{2} - x + \frac{1}{2}x^2 \right) dx \\&= \left[ \frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3 \right]_0^1 \\&= \frac{1}{6}.\end{aligned}$$

# Change of Variables in Double Integrals

## Example

### EXAMPLE

Express the iterated integral  $I = \int_0^1 dy \int_y^1 dz \int_0^z f(x, y, z) dx$  as a triple integral, and sketch the region over which it is taken. Reiterate the integral in such a way that the integrations are performed in the order: first  $y$ , then  $z$ , then  $x$  (i.e., the opposite order to the given iteration).

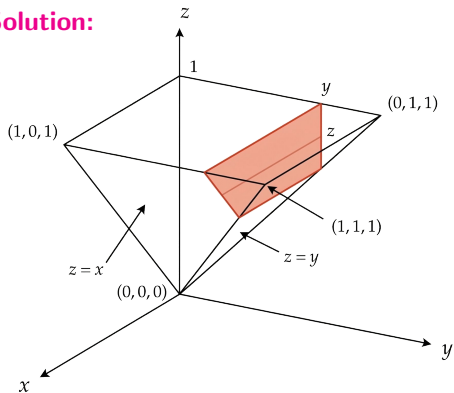
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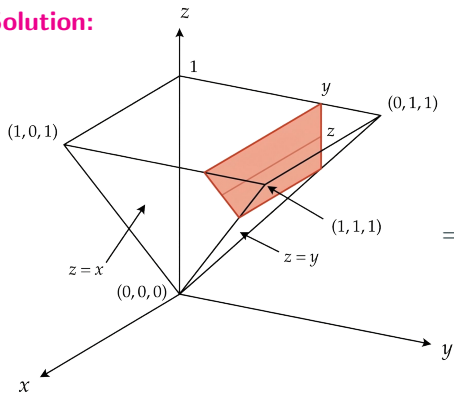
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### Solution:



$$\begin{aligned} I &= \iiint_R f(x, y, z) dV. \\ &= \int_0^1 dx \int_x^1 dz \int_0^z f(x, y, z) dy. \end{aligned}$$

## Average Value of a Function in Space

The average value of a function  $F$  over a region  $D$  in space is defined by the formula

$$\text{Average value of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F \, dV.$$

For example, if  $F(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ , then the average value of  $F$  over  $D$  is the average distance of points in  $D$  from the origin. If  $F(x, y, z)$  is the temperature at  $(x, y, z)$  on a solid that occupies a region  $D$  in space, then the average value of  $F$  over  $D$  is the average temperature of the solid.

# Average Value of a Function in Space

## Example

### EXAMPLE

Find the average value of  $F(x, y, z) = xyz$  throughout the cubical region  $D$  bounded by the coordinate planes and the planes  $x = 2$ ,  $y = 2$ , and  $z = 2$  in the first octant.

# Average Value of a Function in Space

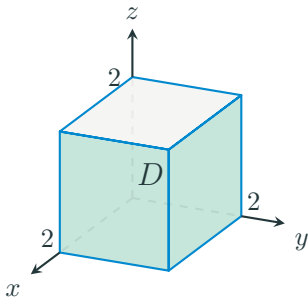
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### Solution:

The volume of the region  $D$  is  $(2)(2)(2) = 8$ .



# Average Value of a Function in Space

## Example

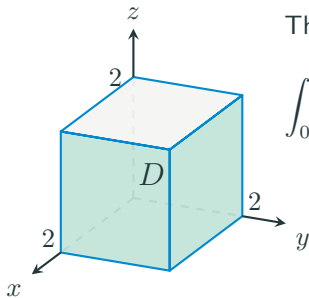
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The value of the integral of  $F$  over the cube is



$$\begin{aligned}\int_0^2 \int_0^2 \int_0^2 xyz \, dx \, dy \, dz &= \int_0^2 \int_0^2 \left[ \frac{x^2}{2} yz \right]_{x=0}^{x=2} dy \, dz \\ &= \int_0^2 \int_0^2 2yz \, dy \, dz \\ &= \int_0^2 [y^2 z]_{y=0}^{y=2} dz \\ &= \int_0^2 4z \, dz = [2z^2]_{z=0}^{z=2} = 8.\end{aligned}$$

# Average Value of a Function in Space

## Example

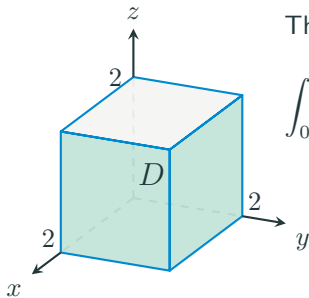
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$$\text{Average value of } xyz \text{ over the cube} = \frac{1}{\text{volume}} \iiint_{\text{cube}} xyz \, dV = \frac{1}{8} \cdot 8 = 1$$